LECTURE NOTES ON GAS DYNAMICS

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Contents

Preface

These are a set of class notes for a gas dynamics/viscous flow course taught to juniors in Aerospace Engineering at the University of Notre Dame during the mid 1990s. The course builds upon foundations laid in an earlier course where the emphasis was on subsonic ideal flows. Consequently, it is expected that the student has some familiarity with many concepts such as material derivatives, control volume analysis, derivation of governing equations, etc. Additionally, first courses in thermodynamics and differential equations are probably necessary. Even a casual reader will find gaps, errors, and inconsistencies. The author welcomes comments and corrections. It is also noted that these notes have been influenced by a variety of standard references, which are sporadically and incompletely noted in the text. Some of the key references which were important in the development of these notes are the texts of Shapiro, Liepmann and Roshko, Anderson, Courant and Friedrichs, Hughes and Brighton, White, Sonntag and Van Wylen, and Zucrow and Hoffman.

At this stage, if anyone outside Notre Dame finds these useful, they are free to make copies. Full information on the course is found at `http://www.nd.edu/~powers/ame.30332`.

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Chapter 1

Introduction

Suggested Reading:

Anderson, Chapter 1: pp. 1-31

1.1 Definitions

The topic of this course is the aerodynamics of compressible and viscous flow.

Where does aerodynamics rest in the taxonomy of mechanics?

Aerodynamics–a branch of dynamics that deals with the motion of *air* and other gaseous fluids and with the *forces* acting on bodies in motion relative to such fluids (*e.g.* airplanes)

We can say that **aerodynamics** is a subset of ($\subset$)

- **fluid dynamics** since air is but one type of fluid, $\subset$
- **fluid mechanics** since dynamics is part of mechanics, $\subset$
- **mechanics** since fluid mechanics is one class of mechanics.

Mechanics–a branch of physical science that deals with forces and the motion of bodies traditionally broken into:

- **kinematics**–study of motion without regard to causality
- **dynamics (kinetics)**–study of forces which give rise to motion

Examples of other subsets of mechanics:

- solid mechanics
- quantum mechanics
- celestial mechanics
- relativistic mechanics
- quantum-electrodynamics (QED)
- magneto-hydrodynamics (MHD)

Recall the definition of a fluid:

Fluid–a material which moves when a shear force is applied.

Recall that solids can, after a small displacement, relax to an equilibrium configuration when a shear force is applied.

Recall also that both *liquids* and *gases* are fluids

The motion of both liquids and gases can be affected by compressibility and shear forces. While shear forces are important for both types of fluids, the influence of compressibility in gases is generally more significant.

The thrust of this class will be to understand how to model the effects of compressibility and shear forces and how this impacts the design of aerospace vehicles.

1.2 Motivating examples

The following two examples serve to illustrate why knowledge of compressibility and shear effects is critical.

1.2.1 Re-entry flows

A range of phenomena are present in the re-entry of a vehicle into the atmosphere. This is an example of an *external* flow. See Figure 1.1.

1.2.1.1 Bow shock wave

- *suddenly* raises density, temperature and pressure of shocked air; consider normal shock in ideal air
  - $\rho_o = 1.16\ kg/m^3 \to \rho_s = 6.64\ kg/m^3$ (over five times as dense!!)
  - $T_o = 300\ K \to T_s = 6,100\ K$ (hot as the sun's surface !!)

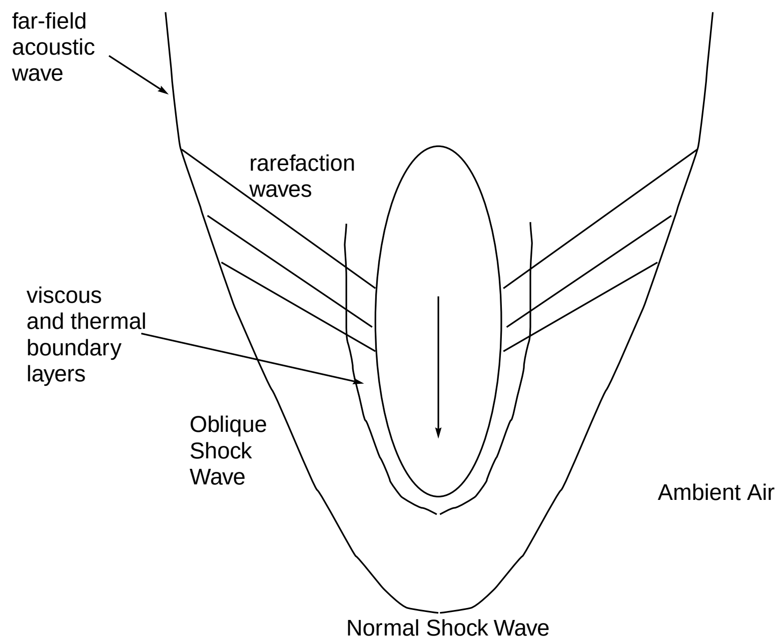

Figure 1.1: Fluid mechanics phenomena in re-entry

- $P_o = 1.0\ atm \rightarrow P_s = 116.5\ atm$ (tremendous force change!!)
- *sudden* transfer of energy from kinetic (ordered) to thermal (random)

- introduces *inviscid* entropy/vorticity layer into post-shocked flow
- normal shock standing off leading edge
- conical oblique shock away from leading edge
- acoustic wave in far field

1.2.1.2 Rarefaction (expansion) wave

- lowers density, temperature, and pressure of air *continuously* and significantly
- interactions with bow shock *weaken* bow shock

1.2.1.3 Momentum boundary layer

- occurs in thin layer near surface where velocity relaxes from freestream to zero to satisfy the *no-slip* condition
- necessary to predict viscous drag forces on body

1.2.1.4 Thermal boundary layer

- as fluid decelerates in momentum boundary layer kinetic energy is converted to thermal energy
- temperature rises can be significant ($> 1,000\ K$)

1.2.1.5 Vibrational relaxation effects

- energy partitioned into *vibrational* modes in addition to translational
- lowers temperature that would otherwise be realized
- important for air above 800 K
- unimportant for monatomic gases

1.2.1.6 Dissociation effects

- effect which happens when multi-atomic molecules split into constituent atoms
- O_2 totally dissociated into O near $4,000\ K$
- N_2 totally dissociated into N near $9,000\ K$
- For $T > 9,000\ K$, ionized plasmas begin to form

Vibrational relaxation, dissociation, and ionization can be accounted for to some extent by introducing a temperature-dependent specific heat $c_v(T)$

1.2.2 Rocket nozzle flows

The same essential ingredients are present in flows through rocket nozzles. This is an example of an *internal* flow, see Figure 1.2

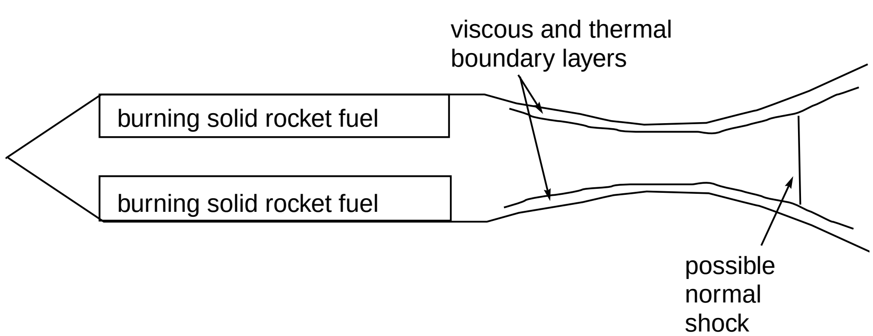

Figure 1.2: Fluid mechanics phenomena in rocket nozzles

Some features:

- well-modelled as one-dimensional flow
- large thrust relies on subsonic to supersonic transition in a converging-diverging nozzle
- away from design conditions normal shocks can exist in nozzle
- viscous and thermal boundary layers must be accounted for in design

1.2.3 Jet engine inlets

The same applies for the internal flow inside a jet engine, see Figure 1.3

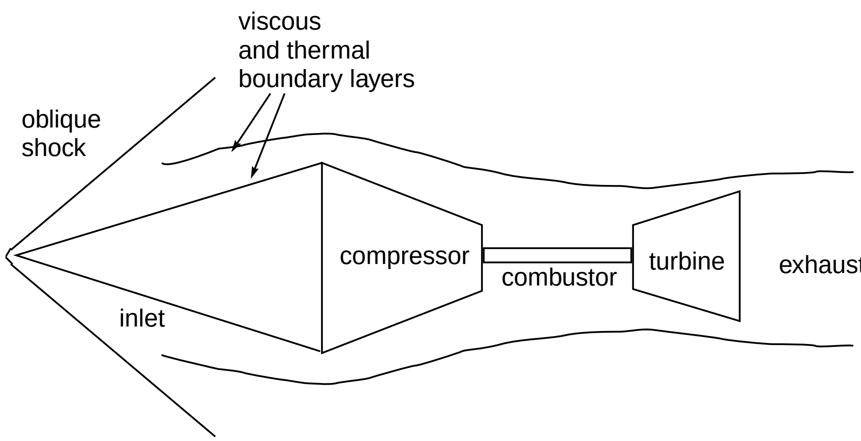

Figure 1.3: Fluid mechanics phenomena in jet engine inlet

Chapter 2

Governing equations

Suggested Reading:

Hughes and Brighton, Chapter 3: pp. 44-64
Liepmann and Roshko, Chapter 7: pp. 178-190, Chapter 13: pp. 305-313, 332-338
Anderson, Chapter 2: pp. 32-44; Chapter 6: pp. 186-205

The equations which govern a wide variety of these flows are the compressible Navier-Stokes equations. In general they are quite complicated and require numerical solution. We will only consider small subsets of these equations in practice, but it is instructive to see them in full glory at the outset.

2.1 Mathematical preliminaries

A few concepts which may be new or need re-emphasis are introduced here.

2.1.1 Vectors and tensors

One way to think of vectors and tensors is as follows:

- first order tensor: **vector**, associates a scalar with any direction in space, column matrix
- second order tensor: **tensor**-associates a vector with any direction in space, two-dimensional matrix
- third order tensor-associates a second order tensor with any direction in space, three-dimensional matrix
- fourth order tensor-...

Here a vector, denoted by boldface, denotes a quantity which can be decomposed as a sum of scalars multiplying orthogonal basis vectors, i.e.:

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \tag{2.1}$$

2.1.2 Gradient, divergence, and material derivatives

The "del" operator, ∇, is as follows:

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{2.2}$$

Recall the definition of the *material* derivative also known as the *substantial* or *total* derivative:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{2.3}$$

where

Example 2.1

Does $\mathbf{v} \cdot \nabla = \nabla \cdot \mathbf{v} = \nabla \mathbf{v}$?

$$\mathbf{v} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{2.4}$$

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2.5}$$

$$\nabla \mathbf{v} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{2.6}$$

So, no.

Here the quantity $\nabla \mathbf{v}$ is an example of a second order tensor. Also

$$\mathbf{v} \cdot \nabla \equiv \mathbf{v} \text{ div} \tag{2.7}$$

$$\nabla \cdot \mathbf{v} \equiv \text{div } \mathbf{v} \tag{2.8}$$

$$\nabla \mathbf{v} \equiv \text{grad } \mathbf{v} \tag{2.9}$$

$$\nabla \phi \equiv \text{grad } \phi \tag{2.10}$$

2.1.3 Conservative and non-conservative forms

If h_i is a column vector of N variables, *e.g.* $h_i = [h_1, h_2, h_3, ...h_N]^T$, and $f_i(h_i)$ $g_i(h_i)$ are a column vectors of N functions of the variables h_i, and all variables are functions of x and t, $h_i = h_i(x,t)$, $f_i(h_i(x,t))$, $g_i(h_i(x,t))$ then a system of partial differential equations is in **conservative form** iff the system can be written as follows:

$$\frac{\partial}{\partial t}h_i + \frac{\partial}{\partial x}\left(f_i(h_i)\right) = g_i(h_i) \tag{2.11}$$

A system not in this form is in **non-conservative form**

2.1.3.1 Conservative form

Advantages

- naturally arises from control volume derivation of governing equations
- clearly exposes groups of terms which are conserved
- easily integrated in certain special cases
- most natural form for deriving normal shock jump equations
- the method of choice for numerical simulations

Disadvantages

- lengthy
- not commonly used
- difficult to see how individual variables change

2.1.3.2 Non-conservative form

Advantages

- compact
- commonly used
- can see how individual variables change

Disadvantages

- often difficult to use to get solutions to problems

- gives rise to artificial instabilities if used in numerical simulation

Example 2.2

Kinematic wave equation

The kinematic wave equation in non-conservative form is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{2.12}$$

This equation has the same mathematical form as **inviscid** equations of gas dynamics which give rise to **discontinuous shock waves**. Thus understanding the solution of this simple equation is very useful in understanding equations with more physical significance.

Since $u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$ the kinematic wave equation in conservative form is as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{2.13}$$

Here $h_i = u, f_i = \frac{u^2}{2}, g_i = 0$.

Consider the special case of a steady state $\frac{\partial}{\partial t} \equiv 0$. Then the conservative form of the equation can be integrated!

$$\frac{d}{dx}\left(\frac{u^2}{2}\right) = 0 \tag{2.14}$$

$$\frac{u^2}{2} = \frac{u_o^2}{2} \tag{2.15}$$

$$u = \pm u_o \tag{2.16}$$

Now $u = u_o$ satisfies the equation and so does $u = -u_o$. These are both smooth solutions. In addition, combinations also satisfy, e.g. $u = u_o, x < 0; u = -u_o, x \geq 0$. This is a discontinuous solution. Also note the solution is *not unique*. This is a consequence of the $u\frac{\partial u}{\partial x}$ non-linearity. This is an example of a type of shock wave. Which solution is achieved generally depends on terms we have neglected, especially unsteady terms.

Example 2.3

Burger's equation

Burger's equation in non-conservative form is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{2.17}$$

This equation has the same mathematical form as **viscous** equations of gas dynamics which give rise to **spatially smeared shock waves**.

Place this in conservative form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial}{\partial x}\frac{\partial u}{\partial x} = 0 \tag{2.18}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) - \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) = 0 \tag{2.19}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} - \nu\frac{\partial u}{\partial x}\right) = 0 \tag{2.20}$$

Here, this equation is not *strictly* in conservative form as it still involves derivatives inside the $\frac{\partial}{\partial x}$ operator.

Consider the special case of a steady state $\frac{\partial}{\partial t} \equiv 0$. Then the conservative form of the equation can be integrated!

$$\frac{d}{dx}\left(\frac{u^2}{2} - \nu\frac{du}{dx}\right) = 0 \tag{2.21}$$

Let $u \to u_o$ as $x \to -\infty$ (consequently $\frac{\partial u}{\partial x} \to 0$ as $x \to -\infty$) and $u(0) = 0$ so

$$\frac{u^2}{2} - \nu\frac{du}{dx} = \frac{u_o^2}{2} \tag{2.22}$$

$$\nu\frac{du}{dx} = \frac{1}{2}\left(u^2 - u_o^2\right) \tag{2.23}$$

$$\frac{du}{u^2 - u_o^2} = \frac{dx}{2\nu} \tag{2.24}$$

$$\int \frac{du}{u_o^2 - u^2} = -\int \frac{dx}{2\nu} \tag{2.25}$$

$$\frac{1}{u_o}\tanh^{-1}\frac{u}{u_o} = -\frac{x}{2\nu} + C \tag{2.26}$$

$$u(x) = u_o \tanh\left(-\frac{u_o}{2\nu}x + Cu_o\right) \tag{2.27}$$

$$u(0) = 0 = u_o \tanh\left(Cu_o\right) \qquad C = 0 \tag{2.28}$$

$$u(x) = u_o \tanh\left(-\frac{u_o}{2\nu}x\right) \tag{2.29}$$

$$\lim_{x\to-\infty} u(x) = u_o \tag{2.30}$$

$$\lim_{x\to\infty} u(x) = -u_o \tag{2.31}$$

Note

- same behavior in far field as kinematic wave equation
- continuous adjustment from u_o to $-u_o$ in a zone of thickness $\frac{2\nu}{u_o}$
- zone thickness $\to 0$ as $\nu \to 0$
- inviscid shock is limiting case of viscously resolved shock

Figure 2.1 gives a plot of the solution to both the kinematic wave equation and Burger's equation.

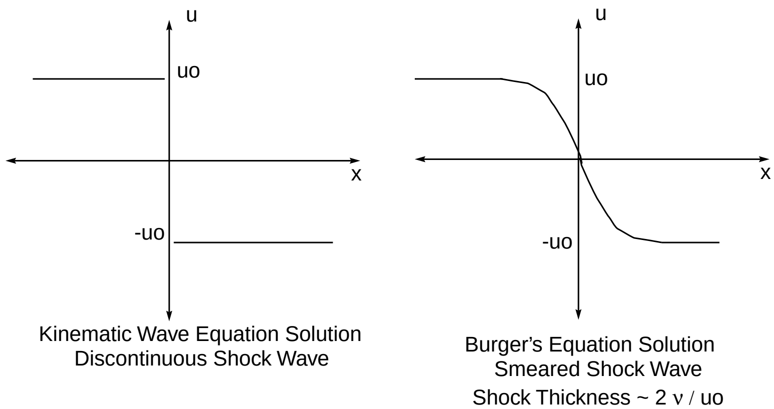

Figure 2.1: Solutions to the kinematic wave equation and Burger's equation

2.2 Summary of full set of compressible viscous equations

A complete set of equations is given below. These are the *compressible Navier-Stokes equations for an isotropic Newtonian fluid with variable properties*

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \qquad [1] \tag{2.32}$$

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P + \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{g} \qquad [3] \tag{2.33}$$

$$\rho\frac{de}{dt} = -\nabla\cdot\mathbf{q} - P\nabla\cdot\mathbf{v} + \boldsymbol{\tau}:\nabla\mathbf{v} \qquad [1] \tag{2.34}$$

$$\boldsymbol{\tau} = \mu\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right) + \lambda\left(\nabla\cdot\mathbf{v}\right)\mathbf{I} \qquad [6] \tag{2.35}$$

$$\mathbf{q} = -k\nabla T \qquad [3] \tag{2.36}$$

$$\mu = \mu\left(\rho, T\right) \qquad [1] \tag{2.37}$$

$$\lambda = \lambda\left(\rho, T\right) \qquad [1] \tag{2.38}$$

$$k = k\left(\rho, T\right) \qquad [1] \tag{2.39}$$

$$P = P\left(\rho, T\right) \qquad [1] \tag{2.40}$$

$$e = e\left(\rho, T\right) \qquad [1] \tag{2.41}$$

The numbers in brackets indicate the number of equations. Here the unknowns are

- ρ–density kg/m^3 (scalar-1 variable)
- $\mathbf{v}$–velocity m/s (vector- 3 variables)
- P–pressure N/m^2 (scalar- 1 variable)
- e–internal energy J/kg (scalar- 1 variable)

- T–temperature K (scalar - 1 variable)
- $\boldsymbol{\tau}$–viscous stress N/m^2 (symmetric tensor - 6 variables)
- $\mathbf{q}$–heat flux vector–W/m^2 (vector - 3 variables)
- μ–first coefficient of viscosity Ns/m^2 (scalar - 1 variable)
- λ–second coefficient of viscosity Ns/m^2 (scalar - 1 variable)
- k–thermal conductivity $W/(m^2K)$ (scalar - 1 variable)

Here $\mathbf{g}$ is the constant gravitational acceleration and $\mathbf{I}$ is the identity matrix. **Total–19 variables**

Points of the exercise

- 19 equations; 19 unknowns
- conservation axioms–postulates (first three equations)
- constitutive relations–material dependent (remaining equations)
- review of vector notation and operations

Exercise: Determine the three Cartesian components of $\nabla\cdot\ \boldsymbol{\tau}$ for a) a compressible Newtonian fluid, and b) an incompressible Newtonian fluid, in which $\nabla \cdot \mathbf{v} = 0$.

This system of equations must be consistent with the second law of thermodynamics. Defining the **entropy** s by the Gibbs relation:

$$T ds = de + P d\left(\frac{1}{\rho}\right) \tag{2.42}$$

$$T\frac{ds}{dt} = \frac{de}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right) \tag{2.43}$$

the second law states:

$$\rho\frac{ds}{dt} \geq -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) \tag{2.44}$$

In practice, this places some simple restrictions on the constitutive relations. It will be sometimes useful to write this in terms of the **specific volume**, $v \equiv 1/\rho$. This can be confused with the y component of velocity but should be clear in context.

2.3 Conservation axioms

Conservation principles are axioms of mechanics and represent statements that cannot be proved. In that they provide predictions which are consistent with empirical observations, they are useful.

2.3.1 Conservation of mass

This principle states that in a **material volume** (a volume which always encompasses the same fluid particles), the mass is constant.

2.3.1.1 Nonconservative form

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{2.45}$$

This can be expanded using the definition of the material derivative to form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial x} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \tag{2.46}$$

2.3.1.2 Conservative form

Using the product rule gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \tag{2.47}$$

The equation essentially says that the net accumulation of mass within a control volume is attributable to the net flux of mass in and out of the control volume. In Gibbs notation this is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.48}$$

2.3.1.3 Incompressible form

Iff the fluid is defined to be incompressible, $d\rho/dt \equiv 0$, the consequence is

$$\nabla \cdot \mathbf{v} = 0, \qquad \text{or} \tag{2.49}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.50}$$

As this course is mainly concerned with *compressible* flow, this will not be often used.

2.3.2 Conservation of linear momenta

This is really Newton's Second Law of Motion $m\mathbf{a} = \sum \mathbf{F}$

2.3.2.1 Nonconservative form

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} \tag{2.51}$$

- ρ: mass/volume

- $\frac{d\mathbf{v}}{dt}$: acceleration
- $-\nabla P, \nabla\cdot\boldsymbol{\tau}$: surface forces/volume
- $\rho\mathbf{g}$: body force/volume

Example 2.4

Expand the term $\nabla\cdot\boldsymbol{\tau}$

$$\nabla\cdot\boldsymbol{\tau} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} \\ \frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} \\ \frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} \end{pmatrix}^T \tag{2.52}$$

This is a **vector** equation as there are three components of momenta. Let's consider the x momentum equation for example.

$$\rho\frac{du}{dt} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x \tag{2.53}$$

Now expand the material derivative:

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x \tag{2.54}$$

Equivalent equations exist for y and z linear momentum:

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y \tag{2.55}$$

$$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_z \tag{2.56}$$

2.3.2.2 Conservative form

Multiply the mass conservation principle by u so that it has the same units as the momentum equation and add to the x momentum equation:

$$u\frac{\partial \rho}{\partial t} + u\frac{\partial(\rho u)}{\partial x} + u\frac{\partial(\rho v)}{\partial y} + u\frac{\partial(\rho w)}{\partial z} = 0 \tag{2.57}$$

$$+\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x \tag{2.58}$$

Using the product rule, this yields:

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho vu)}{\partial y}+\frac{\partial(\rho wu)}{\partial z}=-\frac{\partial P}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\rho g_x \tag{2.59}$$

The extension to y and z momenta is straightforward:

$$\frac{\partial(\rho v)}{\partial t}+\frac{\partial(\rho uv)}{\partial x}+\frac{\partial(\rho vv)}{\partial y}+\frac{\partial(\rho wv)}{\partial z}=-\frac{\partial P}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+\rho g_y \tag{2.60}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho vw)}{\partial y}+\frac{\partial(\rho ww)}{\partial z}=-\frac{\partial P}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}+\rho g_z \tag{2.61}$$

In vector form this is written as follows:

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla\cdot(\rho\mathbf{v}\mathbf{v})=-\nabla P+\nabla\cdot\boldsymbol{\tau}+\rho\mathbf{g} \tag{2.62}$$

As with the mass equation, the time derivative can be interpreted as the *accumulation* of linear momenta within a control volume and the divergence term can be interpreted as the *flux* of linear momenta into the control volume. The accumulation and flux terms are balanced by *forces*, both surface and body.

2.3.3 Conservation of energy

This principle really is the first law of thermodynamics, which states the change in internal energy of a body is equal to the heat added to the body minus the work done by the body;

$$\hat{E}_2-\hat{E}_1=Q_{12}-W_{12} \tag{2.63}$$

The $\hat{E}$ here includes both internal energy and kinetic energy and is written for an extensive system:

$$\hat{E}=\rho V\left(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) \tag{2.64}$$

2.3.3.1 Nonconservative form

The equation we started with (which is in non-conservative form)

$$\rho\frac{de}{dt}=-\nabla\cdot\mathbf{q}-P\nabla\cdot\mathbf{v}+\boldsymbol{\tau}:\nabla\mathbf{v} \tag{2.65}$$

is simply a careful expression of the simple idea $de=dq-dw$ with attention paid to sign conventions, etc.

- $\rho\frac{de}{dt}$: change in internal energy /volume
- $-\nabla\cdot\mathbf{q}$: net heat transfer into fluid/volume
- $P\nabla\cdot\mathbf{v}$: net work done by fluid due to pressure force/volume (force $\times$ deformation)
- $-\boldsymbol{\tau}:\nabla\mathbf{v}$: net work done by fluid due to viscous force/volume (force $\times$ deformation)

2.3.3.2 Mechanical energy

Taking the dot product of the velocity $\mathbf{v}$ with the linear momentum principle yields the mechanical energy equation (here expressed in conservative form):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho\left(\mathbf{v}\cdot\mathbf{v}\right)\right)+\nabla\cdot\left(\frac{1}{2}\rho\mathbf{v}\left(\mathbf{v}\cdot\mathbf{v}\right)\right)=-\mathbf{v}\cdot\nabla P+\mathbf{v}\cdot\left(\nabla\cdot\boldsymbol{\tau}\right)+\rho\mathbf{v}\cdot\mathbf{g} \tag{2.66}$$

This can be interpreted as saying the kinetic energy (or mechanical energy) changes due to

- motion in the direction of a force imbalance
  - $-\mathbf{v}\cdot\nabla P$
  - $\mathbf{v}\cdot(\nabla\cdot\boldsymbol{\tau})$
- motion in the direction of a body force

Exercise: Add the product of the mass equation and $u^2/2$ to the product of u and the one dimensional linear momentum equation:

$$u\left(\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}\right)=u\left(-\frac{\partial P}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\rho g_x\right) \tag{2.67}$$

to form the conservative form of the one-dimensional mechanical energy equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}\rho u^3\right)=-u\frac{\partial P}{\partial x}+u\frac{\partial \tau_{xx}}{\partial x}+\rho u g_x \tag{2.68}$$

2.3.3.3 Conservative form

When we multiply the mass equation by e, we get

$$e\frac{\partial \rho}{\partial t}+e\frac{\partial(\rho u)}{\partial x}+e\frac{\partial(\rho v)}{\partial y}+e\frac{\partial(\rho w)}{\partial z}=0 \tag{2.69}$$

Adding this to the nonconservative energy equation gives

$$\frac{\partial}{\partial t}(\rho e)+\nabla\cdot(\rho\mathbf{v}e)=-\nabla\cdot\mathbf{q}-P\nabla\cdot\mathbf{v}+\boldsymbol{\tau}:\nabla\mathbf{v} \tag{2.70}$$

Adding to this the mechanical energy equation gives the conservative form of the energy equation:

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\right)+\nabla\cdot\left(\rho\mathbf{v}\left(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\right)=-\nabla\cdot\mathbf{q}-\nabla\cdot(P\mathbf{v})+\nabla\cdot(\boldsymbol{\tau}\cdot\mathbf{v})+\rho\mathbf{v}\cdot\mathbf{g} \tag{2.71}$$

which is often written as

$$\frac{\partial}{\partial t}\left(\rho\left(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\right)+\nabla\cdot\left(\rho\mathbf{v}\left(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}+\frac{P}{\rho}\right)\right)=-\nabla\cdot\mathbf{q}+\nabla\cdot(\boldsymbol{\tau}\cdot\mathbf{v})+\rho\mathbf{v}\cdot\mathbf{g} \tag{2.72}$$

2.3.3.4 Energy equation in terms of entropy

Recall the Gibbs relation which defines entropy s:

$$T\frac{ds}{dt} = \frac{de}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{de}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt} \tag{2.73}$$

so

$$\rho\frac{de}{dt} = \rho T\frac{ds}{dt} + \frac{P}{\rho}\frac{d\rho}{dt} \tag{2.74}$$

also from the conservation of mass

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{d\rho}{dt} \tag{2.75}$$

Substitute into nonconservative energy equation:

$$\rho T\frac{ds}{dt} + \frac{P}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \mathbf{q} + \frac{P}{\rho}\frac{d\rho}{dt} + \boldsymbol{\tau} : \nabla\mathbf{v} \tag{2.76}$$

Solve for entropy change:

$$\rho\frac{ds}{dt} = -\frac{1}{T}\nabla \cdot \mathbf{q} + \frac{1}{T}\boldsymbol{\tau} : \nabla\mathbf{v} \tag{2.77}$$

Two effects change entropy:

- heat transfer
- viscous work

Note the work of the pressure force does not change entropy; it is *reversible* work.

If there are no viscous and heat transfer effects, there is no mechanism for entropy change; $ds/dt = 0$; the flow is **isentropic.**

2.3.4 Entropy inequality

The first law can be used to reduce the second law to a very simple form. Starting with

$$\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) = \frac{1}{T}\nabla \cdot \mathbf{q} - \frac{\mathbf{q}}{T^2} \cdot \nabla T \tag{2.78}$$

so

$$-\frac{1}{T}\nabla \cdot \mathbf{q} = -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{\mathbf{q}}{T^2} \cdot \nabla T \tag{2.79}$$

Substitute into the first law:

$$\rho\frac{ds}{dt} = -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{\mathbf{q}}{T^2} \cdot \nabla T + \frac{1}{T}\boldsymbol{\tau} : \nabla\mathbf{v} \tag{2.80}$$

Recall the second law of thermodynamics:

$$\rho \frac{ds}{dt} \geq -\nabla \cdot \left(\frac{\mathbf{q}}{T}\right) \tag{2.81}$$

Substituting the first law into the second law thus yields:

$$-\frac{\mathbf{q}}{T^2} \cdot \nabla T + \frac{1}{T} \boldsymbol{\tau} : \nabla \mathbf{v} \geq 0 \tag{2.82}$$

Our constitutive theory for $\mathbf{q}$ and $\boldsymbol{\tau}$ must be constructed to be constructed so as not to violate the second law.

Exercise: Beginning with the unsteady, two-dimensional, compressible Navier-Stokes equations with no body force in conservative form (below), show all steps necessary to reduce these to the following non-conservative form.

Conservative form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + P - \tau_{xx}) + \frac{\partial}{\partial y}(\rho u v - \tau_{yx}) = 0$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho v u - \tau_{xy}) + \frac{\partial}{\partial y}(\rho v v + P - \tau_{yy}) = 0$$

$$\frac{\partial}{\partial t}\left(\rho\left(e + \frac{1}{2}\left(u^2 + v^2\right)\right)\right)$$
$$+\frac{\partial}{\partial x}\left(\rho u\left(e + \frac{1}{2}\left(u^2 + v^2\right) + \frac{P}{\rho}\right) - (u\tau_{xx} + v\tau_{xy}) + q_x\right)$$
$$+\frac{\partial}{\partial y}\left(\rho v\left(e + \frac{1}{2}\left(u^2 + v^2\right) + \frac{P}{\rho}\right) - (u\tau_{yx} + v\tau_{yy}) + q_y\right) = 0$$

Non-conservative form

$$\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

$$\rho\left(\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y}\right)$$
$$= -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right)$$
$$-P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$
$$+\tau_{xx}\frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{yy}\frac{\partial v}{\partial y}$$

2.4 Constitutive relations

These are determined from experiments and provide sometimes good and sometimes crude models for microstructurally based phenomena.

2.4.1 Stress-strain rate relationship for Newtonian fluids

Perform the experiment described in Figure 2.2.

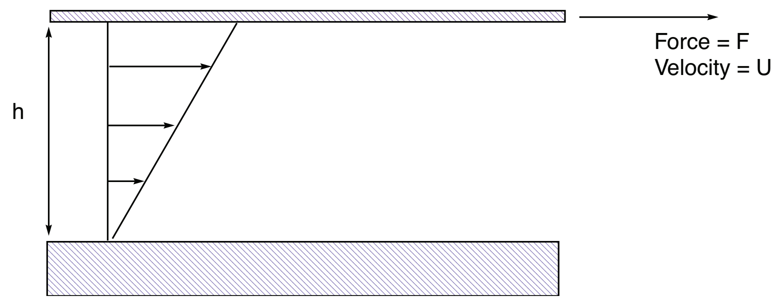

Figure 2.2: Schematic of experiment to determine stress-strain-rate relationship

The following results are obtained, Figure 2.3:

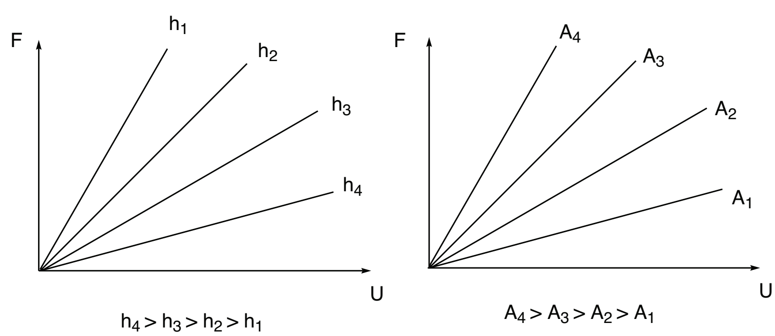

Figure 2.3: Force (N) vs. velocity (m/s)

Note for constant plate velocity U

- small gap width h gives large force F
- large cross-sectional area A gives large force F

When scaled by h and A, for a single fluid, the curve collapses to a single curve, Figure 2.4:

The **viscosity** is defined as the ratio of the applied stress $\tau_{yx} = F/A$ to the strain rate $\frac{\partial u}{\partial y}$.

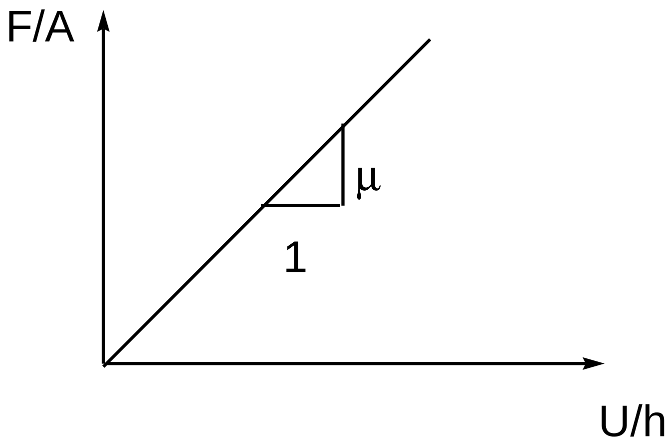

Figure 2.4: Stress (N/m^2) vs. strain rate $(1/s)$

$$\mu \equiv \frac{\tau_{yx}}{\frac{\partial u}{\partial y}} \tag{2.83}$$

Here the first subscript indicates the face on which the force is acting, here the y face. The second subscript indicates the direction in which the force takes, here the x direction. In general viscous stress is a **tensor** quantity. In full detail it is as follows:

$$\boldsymbol{\tau} = \mu \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} \end{bmatrix}$$
$$+\lambda \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} & 0 & 0 \\ 0 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} & 0 \\ 0 & 0 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \end{bmatrix} \tag{2.84}$$

This is simply an expanded form of that written originally:

$$\boldsymbol{\tau} = \mu \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right) + \lambda \left(\nabla \cdot \mathbf{v}\right) \mathbf{I} \tag{2.85}$$

Here λ is the **second coefficient of viscosity**. It is irrelevant in incompressible flows and notoriously difficult to measure in compressible flows. It has been the source of controversy for over 150 years. Commonly, and only for convenience, people take *Stokes' Assumption*:

$$\lambda \equiv -\frac{2}{3}\mu \tag{2.86}$$

It can be shown that this results in the mean mechanical stress being equivalent to the thermodynamic pressure.

It can also be shown that the second law is satisfied if

$$\mu \geq 0 \qquad \text{and} \qquad \lambda \geq -\frac{2}{3}\mu \tag{2.87}$$

Example 2.5
Couette Flow

Use the linear momentum principle and the constitutive theory to show the velocity profile between two plates is linear. The lower plate at $y = 0$ is stationary; the upper plate at $y = h$ is moving at velocity U. Assume $\mathbf{v} = u(y)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. Assume there is no imposed pressure gradient or body force. Assume constant viscosity μ. Since $u = u(y)$, $v = 0, w = 0$, there is no fluid acceleration.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = 0 + 0 + 0 + 0 = 0 \tag{2.88}$$

Since no pressure gradient or body force the linear momentum principle is simply

$$0 = \frac{\partial \tau_{yx}}{\partial y} \tag{2.89}$$

With the Newtonian fluid

$$0 = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{2.90}$$

With constant μ and $u = u(y)$ we have:

$$\mu \frac{d^2 u}{dx^2} = 0 \tag{2.91}$$

Integrating we find

$$u = Ay + B \tag{2.92}$$

Use the boundary conditions at $y = 0$ and $y = h$ to give A and B:

$$A = 0, \qquad B = \frac{U}{h} \tag{2.93}$$

so

$$u(y) = \frac{U}{h}y \tag{2.94}$$

Example 2.6
Poiseuille Flow

Consider flow between a slot separated by two plates, the lower at $y = 0$, the upper at $y = h$, both plates stationary. The flow is driven by a pressure difference. At $x = 0$, $P = P_o$; at $x = L$, $P = P_1$. The fluid has constant viscosity μ. Assuming the flow is steady, there is no body force, pressure varies only with x, and that the velocity is only in the x direction and only a function of y; i.e. $\mathbf{v} = \mathrm{u(y)}\ \mathbf{i}$, find the velocity profile $u(y)$ parameterized by P_o, P_1, h, and μ.

As before there is no acceleration and the x momentum equation reduces to:

$$0 = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{2.95}$$

First let's find the pressure field; take $\partial/\partial x$:

$$0 = -\frac{\partial^2 P}{\partial x^2} + \mu \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right) \tag{2.96}$$

$$\text{changing order of differentiation:} \qquad 0 = -\frac{\partial^2 P}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) \tag{2.97}$$

$$0 = -\frac{\partial^2 P}{\partial x^2} = -\frac{d^2 P}{dx^2} \tag{2.98}$$

$$\frac{dP}{dx} = A \tag{2.99}$$

$$P = Ax + B \tag{2.100}$$

$$\text{apply boundary conditions :} \qquad P(0) = P_o \qquad P(L) = P_1 \tag{2.101}$$

$$P(x) = P_o + (P_1 - P_o)\frac{x}{L} \tag{2.102}$$

$$\text{so} \qquad \frac{dP}{dx} = \frac{(P_1 - P_o)}{L} \tag{2.103}$$

$$\text{substitute into momentum:} \qquad 0 = -\frac{(P_1 - P_o)}{L} + \mu \frac{d^2 u}{dy^2} \tag{2.104}$$

$$\frac{d^2 u}{dy^2} = \frac{(P_1 - P_o)}{\mu L} \tag{2.105}$$

$$\frac{du}{dy} = \frac{(P_1 - P_o)}{\mu L} y + C_1 \tag{2.106}$$

$$u(y) = \frac{(P_1 - P_o)}{2\mu L} y^2 + C_1 y + C_2 \tag{2.107}$$

$$\text{boundary conditions:} \qquad u(0) = 0 = C_2 \tag{2.108}$$

$$u(h) = 0 = \frac{(P_1 - P_o)}{2\mu L} h^2 + C_1 h + 0 \tag{2.109}$$

$$C_1 = -\frac{(P_1 - P_o)}{2\mu L} h \tag{2.110}$$

$$u(y) = \frac{(P_1 - P_o)}{2\mu L}\left(y^2 - yh\right) \tag{2.111}$$

$$\text{wall shear:} \qquad \frac{du}{dy} = \frac{(P_1 - P_o)}{2\mu L}(2y - h) \tag{2.112}$$

$$\tau_{wall} = \mu \left.\frac{du}{dy}\right|_{y=0} = -h\frac{(P_1 - P_o)}{2L} \tag{2.113}$$

Exercise: Consider flow between a slot separated by two plates, the lower at $y = 0$, the upper at $y = h$, with the bottom plate stationary and the upper plate moving at velocity

U. The flow is driven by a pressure difference *and* the motion of the upper plate. At $x = 0$, $P = P_o$; at $x = L$, $P = P_1$. The fluid has constant viscosity μ. Assuming the flow is steady, there is no body force, pressure varies only with x, and that the velocity is only in the x direction and only a function of y; i.e. $\mathbf{v} = u(y)\mathbf{i}$, a) find the velocity profile $u(y)$ parameterized by P_o, P_1, h, U and μ; b) Find U such that there is no net mass flux between the plates.

2.4.2 Fourier's law for heat conduction

It is observed in experiment that heat moves from regions of high temperature to low temperature Perform the experiment described in Figure 2.5.

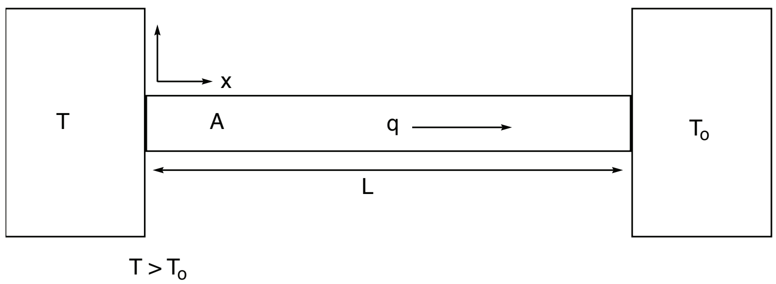

Figure 2.5: Schematic of experiment to determine thermal conductivity

The following results are obtained, Figure 2.6:

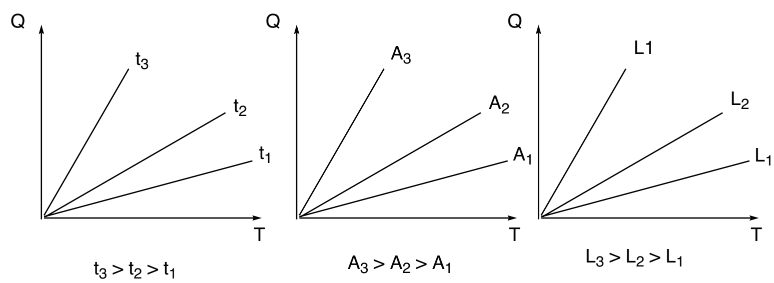

Figure 2.6: Heat transferred (J) vs. temperature (K)

Note for constant temperature of the high temperature reservoir T

- large time of heat transfer t gives large heat transfer Q
- large cross-sectional area A gives large heat transfer Q
- small length L gives large heat transfer Q

When scaled by L, t, and A, for a single fluid, the curve collapses to a single curve, Figure 2.7:

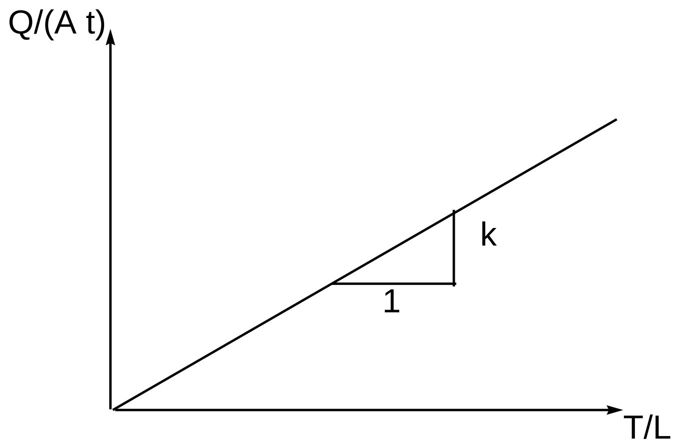

Figure 2.7: heat flux vs. temperature gradient

The **thermal conductivity** is defined as the ratio of the flux of heat transfer $q_x \sim Q/(At)$ to the temperature gradient $-\frac{\partial T}{\partial x} \sim T/L$.

$$k \equiv \frac{q_x}{-\frac{\partial T}{\partial x}} \tag{2.114}$$

so

$$q_x = -k\frac{\partial T}{\partial x} \tag{2.115}$$

or in vector notation:

$$\mathbf{q} = -k\nabla T \tag{2.116}$$

Note with this form, the contribution from heat transfer to the entropy production is guaranteed positive if $k \geq 0$.

$$k\frac{\nabla T \cdot \nabla T}{T^2} + \frac{1}{T}\boldsymbol{\tau} : \nabla \mathbf{v} \geq 0 \tag{2.117}$$

2.4.3 Variable first coefficient of viscosity, μ

In general the first coefficient of viscosity μ is a thermodynamic property which is a strong function of temperature and a weak function of pressure.

2.4.3.1 Typical values of μ for air and water

- air at 300 K, 1 atm : 18.46×10^{-6} $(Ns)/m^2$
- air at 400 K, 1 atm : 23.01×10^{-6} $(Ns)/m^2$
- liquid water at 300 K, 1 atm : 855×10^{-6} $(Ns)/m^2$

- liquid water at 400 $K, 1\ atm$: $217 \times 10^{-6}\ (Ns)/m^2$

Note

- viscosity of air an order of magnitude less than water
- $\frac{\partial \mu}{\partial T} > 0$ for air, and gases in general
- $\frac{\partial \mu}{\partial T} < 0$ for water, and liquids in general

2.4.3.2 Common models for μ

- constant property: $\mu = \mu_o$
- kinetic theory estimate for high temperature gas: $\mu(T) = \mu_o \sqrt{\frac{T}{T_o}}$
- empirical data

2.4.4 Variable second coefficient of viscosity, λ

Very little data for any material exists for the second coefficient of viscosity. It only plays a role in compressible viscous flows, which are typically very high speed. Some estimates:

- Stokes' hypothesis: $\lambda = -\frac{2}{3}\mu$, *may* be correct for monatomic gases
- may be inferred from attenuation rates of sound waves
- perhaps may be inferred from shock wave thicknesses

2.4.5 Variable thermal conductivity, k

In general thermal conductivity k is a thermodynamic property which is a strong function of temperature and a weak function of pressure.

2.4.5.1 Typical values of k for air and water

- air at 300 $K, 1\ atm$: $26.3 \times 10^{-3}\ W/(mK)$
- air at 400 $K, 1\ atm$: $33.8 \times 10^{-3}\ W/(mK)$
- liquid water at 300 $K, 1\ atm$: $613 \times 10^{-3}\ W/(mK)$
- liquid water at 400 $K, 1\ atm$: $688 \times 10^{-3}\ W/(mK)$ (the liquid here is supersaturated)

Note

- conductivity of air is one order of magnitude less than water
- $\frac{\partial k}{\partial T} > 0$ for air, and gases in general
- $\frac{\partial k}{\partial T} > 0$ for water in this range, generalization difficult

2.4.5.2 Common models for k

- constant property: $k = k_o$
- kinetic theory estimate for high temperature gas: $k(T) = k_o\sqrt{\frac{T}{T_o}}$
- empirical data

Exercise: Consider one-dimensional steady heat conduction in a fluid at rest. At $x = 0\ m$ at constant heat flux is applied $q_x = 10\ W/m^2$. At $x = 1\ m$, the temperature is held constant at $300\ K$. Find $T(y)$, $T(0)$ and $q_x(1)$ for

- liquid water with $k = 613 \times 10^{-3}\ W/(mK)$
- air with $k = 26.3 \times 10^{-3}\ W/(mK)$
- air with $k = \left(26.3 \times 10^{-3}\sqrt{\frac{T}{300}}\right)\ W/(mK)$

2.4.6 Thermal equation of state

2.4.6.1 Description

- determined in static experiments
- gives P as a function of ρ and T

2.4.6.2 Typical models

- ideal gas: $P = \rho RT$
- first virial: $P = \rho RT\left(1 + b_1\rho\right)$
- general virial: $P = \rho RT\left(1 + b_1\rho + b_2\rho^2 + ...\right)$
- van der Waals: $P = RT\left(1/\rho - b\right)^{-1} - a\rho^2$

2.4.7 Caloric equation of state

2.4.7.1 Description

- determined in experiments
- gives e as function of ρ and T in general
- arbitrary constant appears
- must also be thermodynamically consistent via relation to be discussed later:

$$de = c_v\,(T)\,dT - \left(\frac{1}{\rho^2}\right)\left(T\,\frac{\partial P}{\partial T}\bigg|_\rho - P\right)d\rho \tag{2.118}$$

With knowledge of $c_v(T)$ and $P(\rho, T)$, the above can be integrated to find e.

2.4.7.2 Typical models

- consistent with ideal gas:
  - constant specific heat: $e(T) = c_{vo}\,(T - T_o) + e_o$
  - temperature dependent specific heat: $e(T) = \int_{T_o}^{T} c_v(\hat{T})d\hat{T} + e_o$
- consistent with first virial: $e(T) = \int_{T_o}^{T} c_v(\hat{T})d\hat{T} + e_o$
- consistent with van der Waals: $e(\rho, T) = \int_{T_o}^{T} c_v(\hat{T})d\hat{T} + -a\,(\rho - \rho_o) + e_o$

2.5 Special cases of governing equations

The governing equations are often expressed in more simple forms in common limits. Some are listed here.

2.5.1 One-dimensional equations

Most of the mystery of vector notation is removed in the one-dimensional limit where $v = w = 0$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$; additionally we adopt Stokes assumption $\lambda = -(2/3)\mu$:

$$\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x}\right) + \rho\frac{\partial u}{\partial x} = 0 \tag{2.119}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\frac{4}{3}\mu\frac{\partial u}{\partial x}\right) + \rho g_x \tag{2.120}$$

$$\rho\left(\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - P\frac{\partial u}{\partial x} + \frac{4}{3}\mu\left(\frac{\partial u}{\partial x}\right)^2 \tag{2.121}$$

$$\mu = \mu\,(\rho, T) \tag{2.122}$$

$$k = k\,(\rho, T) \tag{2.123}$$

$$P = P\,(\rho, T) \tag{2.124}$$

$$e = e\,(\rho, T) \tag{2.125}$$

note: 7 equations, 7 unknowns: $(\rho, u, P, e, T, \mu, k)$

2.5.2 Euler equations

When viscous stresses and heat conduction neglected, the Euler equations are obtained.

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \tag{2.126}$$

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P \tag{2.127}$$

$$\frac{de}{dt} = -P\frac{d}{dt}\left(\frac{1}{\rho}\right) \tag{2.128}$$

$$e = e\left(P,\rho\right) \tag{2.129}$$

Note:

- 6 equations, 6 unknowns (ρ, u, v, w, P, e)
- body force neglected-usually unimportant in this limit
- easy to show this is *isentropic* flow; energy change is all due to reversible Pdv work

Exercise: Write the one-dimensional Euler equations in a) non-conservative form, b) conservative form. Show all steps which lead from one form to the other.

2.5.3 Incompressible Navier-Stokes equations

If we take, ρ, k, μ, c_p to be constant for an ideal gas and neglect viscous dissipation which is *usually* small in such cases:

$$\nabla\cdot\mathbf{v} = 0 \tag{2.130}$$

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla P + \mu\nabla^2\mathbf{v} \tag{2.131}$$

$$\rho c_p\frac{dT}{dt} = k\nabla^2 T \tag{2.132}$$

Note:

- 5 equations, 5 unknowns: (u, v, w, P, T)
- mass and momentum uncoupled from energy
- energy coupled to mass and momentum
- detailed explanation required for use of c_p

Chapter 3

Thermodynamics review

Suggested Reading:

Liepmann and Roshko, Chapter 1: pp. 1-24, 34-38
Shapiro, Chapter 2: pp. 23-44
Anderson, Chapter 1: pp. 12-25

As we have seen from the previous chapter, the subject of thermodynamics is a subset of the topic of viscous compressible flows. It is almost always necessary to consider the thermodynamics as part of a larger coupled system in design. This is in contrast to incompressible aerodynamics which can determine forces independent of the thermodynamics.

3.1 Preliminary mathematical concepts

If

$$z = z(x, y) \tag{3.1}$$

then

$$dz = \left.\frac{\partial z}{\partial x}\right|_y dx + \left.\frac{\partial z}{\partial y}\right|_x dy \tag{3.2}$$

which is of the form

$$dz = Mdx + Ndy \tag{3.3}$$

Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial z}{\partial x} \tag{3.4}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} \tag{3.5}$$

$$\text{thus} \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{3.6}$$

so the implication is that if we are given dz, M, N, we can form z only if the above holds.

3.2 Summary of thermodynamic concepts

- **property:** characterizes the thermodynamics state of the system
  - **extensive:** proportional to system's mass, upper case variable E, S, H
  - **intensive:** independent of system's mass, lower case variable e, s, h, (exceptions T, P)
- **equations of state**: relate properties
- Any intensive thermodynamic property can be expressed as a function of at most **two** other intensive thermodynamic properties (for simple systems)
  - $P = \rho RT$: thermal equation of state for ideal gas
  - $c = \sqrt{\gamma \frac{P}{\rho}}$: sound speed for calorically perfect ideal gas
- **first law**: $d\hat{E} = \delta Q - \delta W$
- **second law**: $dS \geq \delta Q/T$
- **process**: moving from one state to another, in general with accompanying heat transfer and work
- **cycle**: process which returns to initial state
- **reversible work:** $w_{12} = \int_1^2 P dv$
- **reversible heat transfer:** $q_{12} = \int_1^2 T ds$

Figure 3.1 gives a sketch of an isothermal thermodynamic process going from state 1 to state 2. The figure shows a variety of planes, $P - v$, $T - s$, $P - T$, and $v - T$. For ideal gases, 1) isotherms are hyperbolas in the $P - v$ plane: $P = (RT)/v$, 2) isochores are straight lines in the $P - T$ plane: $P = (R/v)T$, with large v giving a small slope, and 3) isobars are straight lines in the $v - T$ plane: $v = (RT)/P$, with large P giving a small slope. The area under the curve in the $P - v$ plane gives the work. The area under the curve in the $T - s$ plane gives the heat transfer. The energy change is given by the difference in the heat transfer and the work. The isochores in the $T - s$ plane are non-trivial. For a calorically perfect ideal gas, they are given by exponential curves.

Figure 3.2 gives a sketch of a thermodynamic cycle. Here we only sketch the $P - v$ and $T - s$ planes, though others could be included. Since it is a cyclic process, there is no net energy change for the cycle and the cyclic work equals the cyclic heat transfer. The enclosed area in the $P - v$ plane, i.e. the net work, equals the enclosed area in the $T - s$ plane, i.e. the net heat transfer. The sketch has the cycle working in the direction which corresponds to an engine. A reversal of the direction would correspond to a refrigerator.

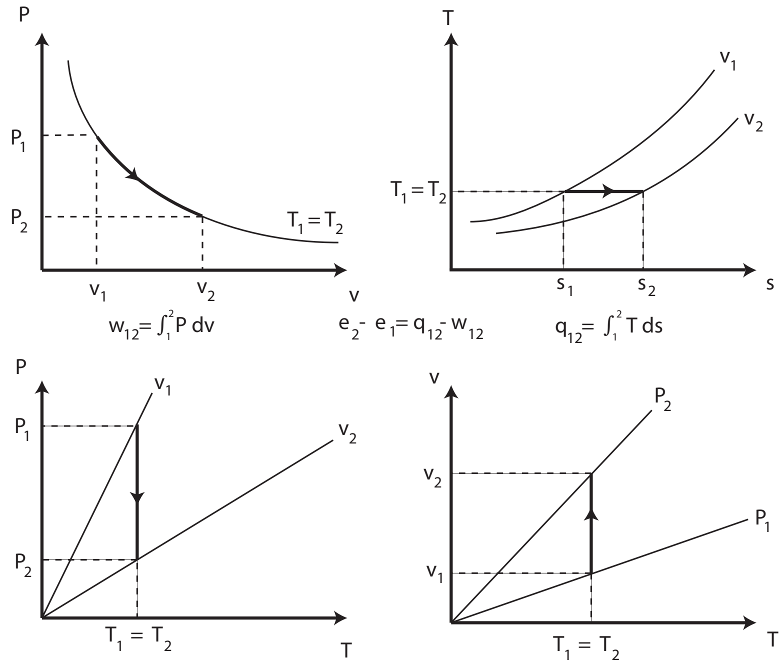

Figure 3.1: Sketch of isothermal thermodynamic process

Example 3.1

Consider the following isobaric process for air, modelled as a calorically perfect ideal gas, from state 1 to state 2. $P_1 = 100\ kPa$, $T_1 = 300\ K$, $T_2 = 400\ K$.

Since the process is isobaric $P = 100\ kPa$ describes a straight line in $P - v$ and $P - T$ planes and $P_2 = P_1 = 100\ kPa$. Since ideal gas, $v - T$ plane:

$$v = \left(\frac{R}{P}\right) T \qquad \text{straight lines!} \tag{3.7}$$

$$v_1 = RT_1/P_1 = \frac{(287\ J/kg/K)\,(300\ K)}{100,000\ Pa} = 0.861\ m^3/kg \tag{3.8}$$

$$v_2 = RT_2/P_2 = \frac{(287\ J/kg/K)\,(400\ K)}{100,000\ Pa} = 1.148\ m^3/kg \tag{3.9}$$

Since calorically perfect:

$$de = c_v dT \tag{3.10}$$

$$\int_{e_2}^{e_1} de = c_v \int_{T_2}^{T_1} dT \tag{3.11}$$

$$e_2 - e_1 = c_v(T_2 - T_1) \tag{3.12}$$

$$= (716.5\ J/kg/K)\,(400\ K - 300\ K) \tag{3.13}$$

$$= 71,650\ J/kg \tag{3.14}$$

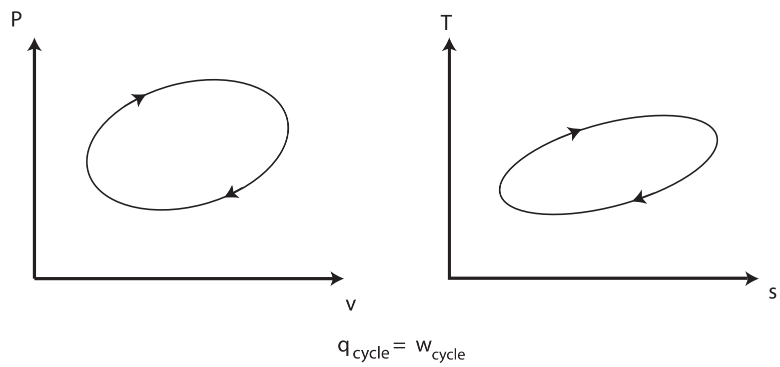

Figure 3.2: Sketch of thermodynamic cycle

also

$$Tds = de + Pdv \tag{3.15}$$

$$Tds = c_v dT + Pdv \tag{3.16}$$

$$\text{from ideal gas :} \qquad v = \frac{RT}{P} : \qquad dv = \frac{R}{P}dT - \frac{RT}{P^2}dP \tag{3.17}$$

$$Tds = c_v dT + RdT - \frac{RT}{P}dP \tag{3.18}$$

$$ds = (c_v + R)\frac{dT}{T} - R\frac{dP}{P} \tag{3.19}$$

$$ds = (c_v + c_p - c_v)\frac{dT}{T} - R\frac{dP}{P} \tag{3.20}$$

$$ds = c_p\frac{dT}{T} - R\frac{dP}{P} \tag{3.21}$$

$$\int_{s_1}^{s_2} ds = c_p \int_{T_1}^{T_2} \frac{dT}{T} - R\int_{P_1}^{P_2} \frac{dP}{P} \tag{3.22}$$

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \tag{3.23}$$

$$s - s_o = c_p \ln\left(\frac{T}{T_o}\right) - R\ln\left(\frac{P}{P_o}\right) \tag{3.24}$$

$$\text{since } P = \text{constant:} \tag{3.25}$$

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) \tag{3.26}$$

$$= (1003.5\ J/kg/K)\ln\left(\frac{400\ K}{300\ K}\right) \tag{3.27}$$

$$= 288.7\ J/kg/K \tag{3.28}$$

$$w_{12} = \int_{v_1}^{v_2} Pdv = P\int_{v_1}^{v_2} dv \tag{3.29}$$

$$= P(v_2 - v_1) \tag{3.30}$$

$$= (100,000\ Pa)(1.148\ m^3/kg - 0.861\ m^3/kg) \tag{3.31}$$
$$= 29,600\ J/kg \tag{3.32}$$

Now

$$de = \delta q - \delta w \tag{3.33}$$
$$\delta q = de + \delta w \tag{3.34}$$
$$q_{12} = (e_2 - e_1) + w_{12} \tag{3.35}$$
$$q_{12} = 71,650\ J/kg + 29,600\ J/kg \tag{3.36}$$
$$q_{12} = 101,250\ J/kg \tag{3.37}$$

Now in this process the gas is heated from 300 K to 400 K. We would expect at a minimum that the surroundings were at 400 K. Let's check for second law satisfaction.

$$s_2 - s_1 \geq \frac{q_{12}}{T_{surr}}? \tag{3.38}$$
$$288.7\ J/kg/K \geq \frac{101,250\ J/kg}{400\ K}? \tag{3.39}$$
$$288.7\ J/kg/K \geq 253.1\ J/kg/K \qquad \text{yes} \tag{3.40}$$

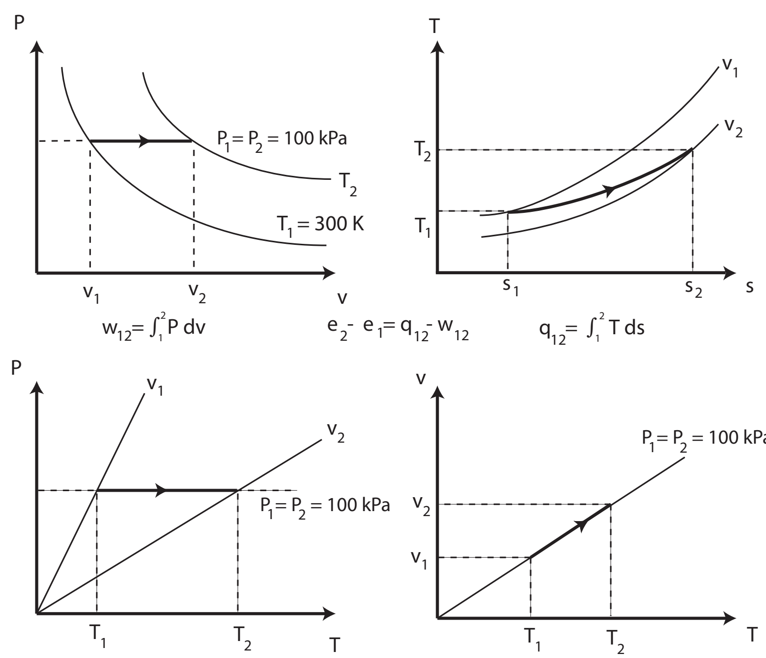

Figure 3.3: Sketch for isobaric example problem

3.3 Maxwell relations and secondary properties

Recall

$$de = Tds - Pd\left(\frac{1}{\rho}\right) \tag{3.41}$$

Since $v \equiv 1/\rho$ we get

$$de = Tds - Pdv \tag{3.42}$$

Now we assume $e = e(s, v)$,

$$de = \left.\frac{\partial e}{\partial s}\right|_v ds + \left.\frac{\partial e}{\partial v}\right|_s dv \tag{3.43}$$

Thus

$$T = \left.\frac{\partial e}{\partial s}\right|_v \qquad P = -\left.\frac{\partial e}{\partial v}\right|_s \tag{3.44}$$

and

$$\left.\frac{\partial T}{\partial v}\right|_s = \frac{\partial^2 e}{\partial v \partial s} \qquad \left.\frac{\partial P}{\partial s}\right|_v = -\frac{\partial^2 e}{\partial s \partial v} \tag{3.45}$$

Thus we get a **Maxwell relation**:

$$\left.\frac{\partial T}{\partial v}\right|_s = -\left.\frac{\partial P}{\partial s}\right|_v \tag{3.46}$$

Define the following properties:

- **enthalpy**: $h \equiv e + pv$
- **Helmholtz free energy**: $a \equiv e - Ts$
- **Gibbs free energy**: $g \equiv h - Ts$

Now with these definitions it is easy to form differential relations using the Gibbs relation as a root.

$$h = e + Pv \tag{3.47}$$
$$dh = de + Pdv + vdP \tag{3.48}$$
$$de = dh - Pdv - vdP \tag{3.49}$$
$$\text{substitute into Gibbs:} \qquad de = Tds - Pdv \tag{3.50}$$
$$dh - Pdv - vdP = Tds - Pdv \tag{3.51}$$
$$dh = Tds + vdP \tag{3.52}$$

So s and P are natural variables for h. Through a very similar process we get the following relationships:

$$\left.\frac{\partial h}{\partial s}\right|_P = T \qquad \left.\frac{\partial h}{\partial P}\right|_s = v \tag{3.53}$$

$$\left.\frac{\partial a}{\partial v}\right|_T = -P \qquad \left.\frac{\partial a}{\partial T}\right|_v = -s \tag{3.54}$$

$$\left.\frac{\partial g}{\partial P}\right|_T = v \qquad \left.\frac{\partial g}{\partial T}\right|_P = -s \tag{3.55}$$

$$\left.\frac{\partial T}{\partial P}\right|_s = \left.\frac{\partial v}{\partial s}\right|_P \qquad \left.\frac{\partial P}{\partial T}\right|_v = \left.\frac{\partial s}{\partial v}\right|_T \qquad \left.\frac{\partial v}{\partial T}\right|_P = -\left.\frac{\partial s}{\partial P}\right|_T \tag{3.56}$$

The following thermodynamic properties are also useful and have formal definitions:

- **specific heat at constant volume**: $c_v \equiv \left.\frac{\partial e}{\partial T}\right|_v$
- **specific heat at constant pressure**: $c_p \equiv \left.\frac{\partial h}{\partial T}\right|_P$
- **ratio of specific heats**: $\gamma \equiv c_p/c_v$
- **sound speed**: $c \equiv \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_s}$
- **adiabatic compressibility**: $\beta_s \equiv -\frac{1}{v}\left.\frac{\partial v}{\partial P}\right|_s$
- **adiabatic bulk modulus**: $B_s \equiv -v\left.\frac{\partial P}{\partial v}\right|_s$

Generic problem: given $P = P(T, v)$, find other properties

3.3.1 Internal energy from thermal equation of state

Find the internal energy $e(T, v)$ for a general material.

$$e = e(T, v) \tag{3.57}$$

$$de = \left.\frac{\partial e}{\partial T}\right|_v dT + \left.\frac{\partial e}{\partial v}\right|_T dv \tag{3.58}$$

$$de = c_v dT + \left.\frac{\partial e}{\partial v}\right|_T dv \tag{3.59}$$

Now from Gibbs,

$$de = T ds - P dv \tag{3.60}$$

$$\frac{de}{dv} = T\frac{ds}{dv} - P \tag{3.61}$$

$$\left.\frac{\partial e}{\partial v}\right|_T = T\left.\frac{\partial s}{\partial v}\right|_T - P \tag{3.62}$$

Substitute from Maxwell relation,

$$\left.\frac{\partial e}{\partial v}\right|_T = T \left.\frac{\partial P}{\partial T}\right|_v - P \tag{3.63}$$

so

$$de = c_v dT + \left(T \left.\frac{\partial P}{\partial T}\right|_v - P \right) dv \tag{3.64}$$

$$\int_{e_o}^{e} d\hat{e} = \int_{T_o}^{T} c_v(\hat{T}) d\hat{T} + \int_{v_o}^{v} \left(\hat{T} \left.\frac{\partial \hat{P}}{\partial \hat{T}}\right|_{\hat{v}} - \hat{P} \right) d\hat{v} \tag{3.65}$$

$$e(T,v) = e_o + \int_{T_o}^{T} c_v(\hat{T}) d\hat{T} + \int_{v_o}^{v} \left(\hat{T} \left.\frac{\partial \hat{P}}{\partial \hat{T}}\right|_{\hat{v}} - \hat{P} \right) d\hat{v} \tag{3.66}$$

Example 3.2

Ideal gas

Find a general expression for $e(T,v)$ if

$$P(T,v) = \frac{RT}{v} \tag{3.67}$$

Proceed as follows:

$$\left.\frac{\partial P}{\partial T}\right|_v = R/v \tag{3.68}$$

$$T \left.\frac{\partial P}{\partial T}\right|_v - P = \frac{RT}{v} - P \tag{3.69}$$

$$= \frac{RT}{v} - \frac{RT}{v} = 0 \tag{3.70}$$

Thus e is

$$e(T) = e_o + \int_{T_o}^{T} c_v(\hat{T}) d\hat{T} \tag{3.71}$$

We also find

$$h = e + Pv = e_o + \int_{T_o}^{T} c_v(\hat{T}) d\hat{T} + Pv \tag{3.72}$$

$$h(T,v) = e_o + \int_{T_o}^{T} c_v(\hat{T}) d\hat{T} + RT \tag{3.73}$$

$$c_p(T,v) =\equiv \left.\frac{\partial h}{\partial T}\right|_P = c_v(T) + R = c_p(T) \tag{3.74}$$

$$R = c_p(T) - c_v(T) \tag{3.75}$$

Iff c_v is a constant then

$$e(T) = e_o + c_v(T - T_o) \tag{3.76}$$

$$h(T) = (e_o + P_o v_o) + c_p(T - T_o) \tag{3.77}$$

$$R = c_p - c_v \tag{3.78}$$

Example 3.3

van der Waals gas

Find a general expression for $e(T, v)$ if

$$P(T, v) = \frac{RT}{v-b} - \frac{a}{v^2} \tag{3.79}$$

Proceed as before:

$$\left.\frac{\partial P}{\partial T}\right|_v = \frac{R}{v-b} \tag{3.80}$$

$$T\left.\frac{\partial P}{\partial T}\right|_v - P = \frac{RT}{v-b} - P \tag{3.81}$$

$$= \frac{RT}{v-b} - \left(\frac{RT}{v-b} - \frac{a}{v^2}\right) = \frac{a}{v^2} \tag{3.82}$$

Thus e is

$$e(T, v) = e_o + \int_{T_o}^{T} c_v(\hat{T})d\hat{T} + \int_{v_o}^{v} \frac{a}{\hat{v}^2}d\hat{v} \tag{3.83}$$

$$= e_o + \int_{T_o}^{T} c_v(\hat{T})d\hat{T} + a\left(\frac{1}{v_o} - \frac{1}{v}\right) \tag{3.84}$$

We also find

$$h = e + Pv = e_o + \int_{T_o}^{T} c_v(\hat{T})d\hat{T} + a\left(\frac{1}{v_o} - \frac{1}{v}\right) + Pv \tag{3.85}$$

$$h(T, v) = e_o + \int_{T_o}^{T} c_v(\hat{T})d\hat{T} + a\left(\frac{1}{v_o} - \frac{1}{v}\right) + \frac{RTv}{v-b} - \frac{a}{v} \tag{3.86}$$

$$\tag{3.87}$$

3.3.2 Sound speed from thermal equation of state

Find the sound speed $c(T, v)$ for a general material.

$$c = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_s} \tag{3.88}$$

$$c^2 = \left.\frac{\partial P}{\partial \rho}\right|_s \tag{3.89}$$

Use Gibbs relation

$$Tds = de + Pdv \tag{3.90}$$
$$\tag{3.91}$$

Substitute earlier relation for de

$$Tds = \left[c_v dT + \left(T \left.\frac{\partial P}{\partial T}\right|_v - P\right) dv\right] + Pdv \tag{3.92}$$

$$Tds = c_v dT + T \left.\frac{\partial P}{\partial T}\right|_v dv \tag{3.93}$$

$$Tds = c_v dT - \frac{T}{\rho^2} \left.\frac{\partial P}{\partial T}\right|_\rho d\rho \tag{3.94}$$

Since $P = P(T, v)$, $P = P(T, \rho)$

$$dP = \left.\frac{\partial P}{\partial T}\right|_\rho dT + \left.\frac{\partial P}{\partial \rho}\right|_T d\rho \tag{3.95}$$

$$dT = \frac{dP - \left.\frac{\partial P}{\partial \rho}\right|_T d\rho}{\left.\frac{\partial P}{\partial T}\right|_\rho} \tag{3.96}$$

Thus substituting for dT

$$Tds = c_v \left(\frac{dP - \left.\frac{\partial P}{\partial \rho}\right|_T d\rho}{\left.\frac{\partial P}{\partial T}\right|_\rho}\right) - \frac{T}{\rho^2} \left.\frac{\partial P}{\partial T}\right|_\rho d\rho \tag{3.97}$$

so grouping terms in dP and $d\rho$ we get

$$Tds = \left(\frac{c_v}{\left.\frac{\partial P}{\partial T}\right|_\rho}\right) dP - \left(c_v \frac{\left.\frac{\partial P}{\partial \rho}\right|_T}{\left.\frac{\partial P}{\partial T}\right|_\rho} + \frac{T}{\rho^2} \left.\frac{\partial P}{\partial T}\right|_\rho\right) d\rho \tag{3.98}$$

SO if $ds \equiv 0$ we obtain

$$\left.\frac{\partial P}{\partial \rho}\right|_s = \frac{1}{c_v} \left.\frac{\partial P}{\partial T}\right|_\rho \left(c_v \frac{\left.\frac{\partial P}{\partial \rho}\right|_T}{\left.\frac{\partial P}{\partial T}\right|_\rho} + \frac{T}{\rho^2} \left.\frac{\partial P}{\partial T}\right|_\rho\right) \tag{3.99}$$

$$= \left.\frac{\partial P}{\partial \rho}\right|_T + \frac{T}{c_v \rho^2} \left(\left.\frac{\partial P}{\partial T}\right|_\rho\right)^2 \tag{3.100}$$

So

$$c(T, \rho) = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_T + \frac{T}{c_v \rho^2} \left(\left.\frac{\partial P}{\partial T}\right|_\rho\right)^2} \tag{3.101}$$

Exercises: Liepmann and Roshko, 1.3 and 1.4, p. 383.

Example 3.4
Ideal gas

Find the sound speed if

$$P(T,\rho) = \rho RT \tag{3.102}$$

The necessary partials are

$$\left.\frac{\partial P}{\partial \rho}\right|_T = RT \qquad \left.\frac{\partial P}{\partial T}\right|_\rho = \rho R \tag{3.103}$$

so

$$c(T,\rho) = \sqrt{RT + \frac{T}{c_v \rho^2}\left(\rho R\right)^2} \tag{3.104}$$

$$= \sqrt{RT + \frac{R^2 T}{c_v}} \tag{3.105}$$

$$= \sqrt{RT\left(1 + \frac{R}{c_v}\right)} \tag{3.106}$$

$$= \sqrt{RT\left(1 + \frac{c_P - c_v}{c_v}\right)} \tag{3.107}$$

$$= \sqrt{RT\left(\frac{c_v + c_P - c_v}{c_v}\right)} \tag{3.108}$$

$$= \sqrt{\gamma RT} \tag{3.109}$$

Sound speed depends on temperature alone for the calorically perfect ideal gas.

Example 3.5
Virial gas

Find the sound speed if

$$P(T,\rho) = \rho RT\left(1 + b_1\rho\right) \tag{3.110}$$

The necessary partials are

$$\left.\frac{\partial P}{\partial \rho}\right|_T = RT + 2b_1\rho RT \qquad \left.\frac{\partial P}{\partial T}\right|_\rho = \rho R\left(1 + b_1\rho\right) \tag{3.111}$$

so

$$c(T,\rho) = \sqrt{RT + 2b_1\rho RT + \frac{T}{c_v\rho^2}\left(\rho R\left(1+b_1\rho\right)\right)^2} \quad (3.112)$$

$$= \sqrt{RT\left(1 + 2b_1\rho + \frac{R}{c_v}(1+b_1\rho)^2\right)} \quad (3.113)$$

Sound speed depends on both temperature **and** density.

Example 3.6

Thermodynamic process with a van der Waals Gas

A van der Waals gas with

$$R = 200\ J/kg/K \quad (3.114)$$

$$a = 150\ Pa\ m^6/kg^2 \quad (3.115)$$

$$b = 0.001\ m^3/kg \quad (3.116)$$

$$c_v = [350 + 0.2(T - 300K)]\ \ J/kg/K \quad (3.117)$$

begins at $T_1 = 300\ K$, $P_1 = 1\times 10^5\ Pa$. It is isothermally compressed to state 2 where $P_2 = 1\times 10^6\ Pa$. It is then isochorically heated to state 3 where $T_3 = 1,000\ K$. Find w_{13}, q_{13}, and $s_3 - s_1$. Assume the surroundings are at $1,000\ K$. Recall

$$P = \frac{RT}{v-b} - \frac{a}{v^2} \quad (3.118)$$

so at state 1

$$100,000 = \frac{200\times 300}{v_1 - 0.001} - \frac{150}{v_1^2} \quad (3.119)$$

or expanding

$$-0.15 + 150v - 60,100v^2 + 100,000v^3 = 0 \quad (3.120)$$

Cubic equation–three solutions:

$$v_1 = 0.598\ m^3/kg \quad (3.121)$$

$$v_1 = 0.00125 - 0.0097i\ m^3/kg \qquad \text{not physical} \quad (3.122)$$

$$v_1 = 0.00125 + 0.0097i\ m^3/kg \qquad \text{not physical} \quad (3.123)$$

Now at state 2 we know P_2 and T_2 so we can determine v_2

$$1,000,000 = \frac{200\times 300}{v_2 - 0.001} - \frac{150}{v_2^2} \quad (3.124)$$

The physical solution is $v_2 = 0.0585\ m^3/kg$. Now at state 3 we know $v_3 = v_2$ and T_3. Determine P_3:

$$P_3 = \frac{200\times 1,000}{0.0585 - 0.001} - \frac{150}{0.0585^2} = 3,478,261 - 43,831 = 3,434,430\ Pa \quad (3.125)$$

Now $w_{13} = w_{12} + w_{23} = \int_1^2 Pdv + \int_2^3 Pdv = \int_1^2 Pdv$ since $2-3$ is at constant volume. So

$$w_{13} = \int_{v_1}^{v_2} \left(\frac{RT}{v-b} - \frac{a}{v^2} \right) dv \tag{3.126}$$

$$= RT_1 \int_{v_1}^{v_2} \frac{dv}{v-b} - a \int_{v_1}^{v_2} \frac{dv}{v^2} \tag{3.127}$$

$$= RT_1 \ln \left(\frac{v_2 - b}{v_1 - b} \right) + a \left(\frac{1}{v_2} - \frac{1}{v_1} \right) \tag{3.128}$$

$$= 200 \times 300 \ln \left(\frac{0.0585 - 0.001}{0.598 - 0.001} \right) + 150 \left(\frac{1}{0.0585} - \frac{1}{0.598} \right) \tag{3.129}$$

$$= -140,408 + 2,313 \tag{3.130}$$

$$= -138,095 \; J/kg = -138 \; kJ/kg \tag{3.131}$$

The gas is compressed, so the work is negative. Since e is a state property:

$$e_3 - e_1 = \int_{T_1}^{T_3} c_v(T) dT + a \left(\frac{1}{v_1} - \frac{1}{v_3} \right) \tag{3.132}$$

Now

$$c_v = 350 + 0.2(T - 300) = 290 + \frac{1}{5} T \tag{3.133}$$

so

$$e_3 - e_1 = \int_{T_1}^{T_3} \left(290 + \frac{1}{5} T \right) dT + a \left(\frac{1}{v_1} - \frac{1}{v_3} \right) \tag{3.134}$$

$$= 290 \left(T_3 - T_1 \right) + \frac{1}{10} \left(T_3^2 - T_1^2 \right) + a \left(\frac{1}{v_1} - \frac{1}{v_3} \right) \tag{3.135}$$

$$290 \left(1,000 - 300 \right) + \frac{1}{10} \left(1,000^2 - 300^2 \right) + 150 \left(\frac{1}{0.598} - \frac{1}{0.0585} \right) \tag{3.136}$$

$$= 203,000 + 91,000 - 2,313 \tag{3.137}$$

$$= 291,687 \; J/kg = 292 \; kJ/kg \tag{3.138}$$

Now from the first law

$$e_3 - e_1 = q_{13} - w_{13} \tag{3.139}$$

$$q_{13} = e_3 - e_1 + w_{13} \tag{3.140}$$

$$q_{13} = 292 - 138 \tag{3.141}$$

$$q_{13} = 154 \; kJ/kg \tag{3.142}$$

The heat transfer is positive as heat was added to the system.

Now find the entropy change. Manipulate the Gibbs equation:

$$Tds = de + Pdv \tag{3.143}$$

$$ds = \frac{1}{T} de + \frac{P}{T} dv \tag{3.144}$$

$$ds = \frac{1}{T} \left(c_v(T) dT + \frac{a}{v^2} dv \right) + \frac{P}{T} dv \tag{3.145}$$

$$ds = \frac{1}{T}\left(c_v(T)dT + \frac{a}{v^2}dv\right) + \frac{1}{T}\left(\frac{RT}{v-b} - \frac{a}{v^2}\right)dv \tag{3.146}$$

$$ds = \frac{c_v(T)}{T}dT + \frac{R}{v-b}dv \tag{3.147}$$

$$s_3 - s_1 = \int_{T_1}^{T_3} \frac{c_v(T)}{T}dT + R\ln\frac{v_3-b}{v_1-b} \tag{3.148}$$

$$= \int_{300}^{1,000}\left(\frac{290}{T} + \frac{1}{5}\right)dT + R\ln\frac{v_3-b}{v_1-b} \tag{3.149}$$

$$= 290\ln\frac{1,000}{300} + \frac{1}{5}(1,000-300) + 200\ln\frac{0.0585-0.001}{0.598-0.001} \tag{3.150}$$

$$= 349 + 140 - 468 \tag{3.151}$$

$$= 21\ \frac{J}{kg\ K} = 0.021\ \frac{kJ}{kg\ K} \tag{3.152}$$

Is the second law satisfied for each portion of the process?

First look at $1 \rightarrow 2$

$$e_2 - e_1 = q_{12} - w_{12} \tag{3.153}$$

$$q_{12} = e_2 - e_1 + w_{12} \tag{3.154}$$

$$q_{12} = \left(\int_{T_1}^{T_2} c_v(T)dT + a\left(\frac{1}{v_1} - \frac{1}{v_2}\right)\right) + \left(RT_1\ln\left(\frac{v_2-b}{v_1-b}\right) + a\left(\frac{1}{v_2} - \frac{1}{v_1}\right)\right) \tag{3.155}$$

$$\tag{3.156}$$

Since $T_1 = T_2$ and canceling the terms in a we get

$$q_{12} = RT_1\ln\left(\frac{v_2-b}{v_1-b}\right) = 200\times 300\ln\left(\frac{0.0585-0.001}{0.598-0.001}\right) = -140,408\ \frac{J}{kg} \tag{3.157}$$

Since isothermal

$$s_2 - s_1 = R\ln\left(\frac{v_2-b}{v_1-b}\right) \tag{3.158}$$

$$= 200\ln\left(\frac{0.0585-0.001}{0.598-0.001}\right) \tag{3.159}$$

$$= -468.0\ \frac{J}{kg\ K} \tag{3.160}$$

Entropy *drops* because heat was transferred *out* of the system.

Check the second law. Note that in this portion of the process in which the heat is transferred out of the system, that the surroundings must have $T_{surr} \leq 300\ K$. For this portion of the process let us take $T_{surr} = 300\ K$.

$$s_2 - s_1 \geq \frac{q_{12}}{T}? \tag{3.161}$$

$$-468.0\ \frac{J}{kg\ K} \geq \frac{-140,408\ \frac{J}{kg}}{300\ K} \tag{3.162}$$

$$-468.0\ \frac{J}{kg\ K} \geq -468.0\ \frac{J}{kg\ K} \qquad \text{ok} \tag{3.163}$$

Next look at $2 \rightarrow 3$

$$q_{23} = e_3 - e_2 + w_{23} \tag{3.164}$$

$$q_{23} = \left(\int_{T_2}^{T_3} c_v(T) dT + a \left(\frac{1}{v_2} - \frac{1}{v_3} \right) \right) + \left(\int_{v_2}^{v_3} P dv \right) \tag{3.165}$$

$$\text{since isochoric} \qquad q_{23} = \int_{T_2}^{T_3} c_v(T) dT \tag{3.166}$$

$$= \int_{300}^{1000} \left(290 + \frac{T}{5} \right) dT = 294,000 \ \frac{J}{K} \tag{3.167}$$

Now look at the entropy change for the isochoric process:

$$s_3 - s_2 = \int_{T_2}^{T_3} \frac{c_v(T)}{T} dT \tag{3.168}$$

$$= \int_{T_2}^{T_3} \left(\frac{290}{T} + \frac{1}{5} \right) dT \tag{3.169}$$

$$= 290 \ln \frac{1,000}{300} + \frac{1}{5}(1,000 - 300) = 489 \ \frac{J}{kg \ K} \tag{3.170}$$

Entropy *rises* because heat transferred *into* system.

In order to transfer heat into the system we must have a different thermal reservoir. This one must have $T_{surr} \geq 1000 \ K$. Assume here that the heat transfer was from a reservoir held at $1,000 \ K$ to assess the influence of the second law.

$$s_3 - s_2 \geq \frac{q_{23}}{T} ? \tag{3.171}$$

$$489 \ \frac{J}{kg \ K} \geq \frac{294,000 \ \frac{J}{kg}}{1,000 \ K} \tag{3.172}$$

$$489 \ \frac{J}{kg \ K} \geq 294 \ \frac{J}{kg \ K} \qquad \text{ok} \tag{3.173}$$

3.4 Canonical equations of state

If we have a single equation of state in a special *canonical* form, we can form both thermal and caloric equations. Since

$$de = T ds - P dv \tag{3.174}$$

$$dh = T ds + v dP \tag{3.175}$$

it is suggested that the form

$$e = e(s, v) \tag{3.176}$$

$$h = h(s, P) \tag{3.177}$$

is useful.

Example 3.7

Canonical Form

If

$$h(s,P) = Kc_pP^{R/c_p}\exp\left(\frac{s}{c_p}\right) + (h_o - c_pT_o) \quad (3.178)$$

derive both thermal and caloric state equations $P(v,T)$ and $e(v,T)$.

Now for our material

$$\left.\frac{\partial h}{\partial s}\right|_P = KP^{R/c_p}\exp\left(\frac{s}{c_p}\right) \quad (3.179)$$

$$\left.\frac{\partial h}{\partial P}\right|_s = KRP^{R/c_p-1}\exp\left(\frac{s}{c_p}\right) \quad (3.180)$$

Now since

$$\left.\frac{\partial h}{\partial s}\right|_P = T \quad (3.181)$$

$$\left.\frac{\partial h}{\partial P}\right|_s = v \quad (3.182)$$

we have

$$T = KP^{R/c_p}\exp\left(\frac{s}{c_p}\right) \quad (3.183)$$

$$v = KRP^{R/c_p-1}\exp\left(\frac{s}{c_p}\right) \quad (3.184)$$

Dividing one by the other gives

$$\frac{T}{v} = \frac{P}{R} \quad (3.185)$$

$$P = \frac{RT}{v} \quad (3.186)$$

Substituting our expression for T into our canonical equation for h we also get

$$h = c_pT + (h_o - c_pT_o) \quad (3.187)$$

$$h = c_p(T - T_o) + h_o \quad (3.188)$$

which is useful in itself. Substituting in for T and T_o

$$h = c_p\left(\frac{Pv}{R} - \frac{P_ov_o}{R}\right) + h_o \quad (3.189)$$

Using $h \equiv e + Pv$ we get

$$e + Pv = c_p\left(\frac{Pv}{R} - \frac{P_ov_o}{R}\right) + e_o + P_ov_o \quad (3.190)$$

so

$$e = \left(\frac{c_p}{R} - 1\right) Pv - \left(\frac{c_p}{R} - 1\right) P_o v_o + e_o \tag{3.191}$$
$$e = \left(\frac{c_p}{R} - 1\right) (Pv - P_o v_o) + e_o \tag{3.192}$$
$$e = \left(\frac{c_p}{R} - 1\right) (RT - RT_o) + e_o \tag{3.193}$$
$$e = (c_p - R)(T - T_o) + e_o \tag{3.194}$$
$$e = [c_p - (c_p - c_v)](T - T_o) + e_o \tag{3.195}$$
$$e = c_v (T - T_o) + e_o \tag{3.196}$$

So **one** canonical equation gives us all the information we need! Oftentimes, it is difficult to do a single experiment to get the canonical form.

Exercise: For a calorically perfect ideal gas, write the Helmholtz free energy and Gibbs free energy in canonical form, i.e. what is $a(T, v)$, $g(P, T)$?

3.5 Isentropic relations

Of particular importance in thermodynamics in general and compressible flow in particular are relations that describe an isentropic process, $s = constant$. Recall the second law.

$$ds \geq \frac{\delta q}{T} \tag{3.197}$$

If the process is **reversible**,

$$ds = \frac{\delta q}{T} \tag{3.198}$$

If the process is **adiabatic**

$$\delta q \equiv 0 \qquad \text{so} \qquad ds = 0 \tag{3.199}$$

So an isentropic process is both adiabatic and reversible. We know from the first law written in terms of entropy that this implies that

$$\mathbf{q} \equiv 0 \tag{3.200}$$
$$\boldsymbol{\tau} \equiv 0 \tag{3.201}$$

In this case the Gibbs relation and the first law reduce to the same expression:

$$de = -P dv \tag{3.202}$$

That is the energy change is all due to reversible pressure volume work.

We would like to develop an expression between two variables for an isentropic process.

With knowledge of $P(T, v)$

- form $e(T, v)$
- eliminate T to form $e(P, v)$
- take derivative and substitute into Gibbs/First Law

$$\left.\frac{\partial e}{\partial P}\right|_v dP + \left.\frac{\partial e}{\partial v}\right|_P dv = -Pdv \tag{3.203}$$

$$\left.\frac{\partial e}{\partial P}\right|_v dP + \left(\left.\frac{\partial e}{\partial v}\right|_P + P\right) dv = 0 \tag{3.204}$$

Integration of this equation gives a relationship between P and v.

Example 3.8

Calorically Perfect Ideal Gas

Find the relationship for a calorically perfect ideal gas which undergoes an isentropic process.

Ideal Gas:

$$Pv = RT \tag{3.205}$$

Calorically Perfect:

$$e = c_v T + e_o \tag{3.206}$$

Thus

$$e = c_v \frac{Pv}{R} + e_o = \frac{c_v}{c_P - c_v} Pv + e_o = \frac{1}{\gamma - 1} Pv + e_o \tag{3.207}$$

Thus the necessary derivatives are

$$\left.\frac{\partial e}{\partial P}\right|_v = \frac{1}{\gamma - 1} v \tag{3.208}$$

$$\left.\frac{\partial e}{\partial v}\right|_P = \frac{1}{\gamma - 1} P \tag{3.209}$$

so substituting into our developed relationship gives

$$\frac{1}{\gamma - 1} v dP + \left(\frac{1}{\gamma - 1} P + P\right) dv = 0 \tag{3.210}$$

$$v dP + \gamma P dv = 0 \tag{3.211}$$

$$\frac{dP}{P} = -\gamma \frac{dv}{v} \tag{3.212}$$

$$\ln \frac{P}{P_o} = -\gamma \ln \frac{v}{v_o} \tag{3.213}$$

$$\ln \frac{P}{P_o} = \ln \left(\frac{v_o}{v}\right)^\gamma \tag{3.214}$$

$$\frac{P}{P_o} = \left(\frac{v_o}{v}\right)^\gamma \tag{3.215}$$

$$Pv^\gamma = P_o v_o^\gamma = \text{constant} \tag{3.216}$$

also using the thermal state equation

$$\frac{P}{P_o} = \frac{\frac{RT}{v}}{\frac{RT_o}{v_o}} = \frac{T}{T_o}\frac{v_o}{v} = \left(\frac{v_o}{v}\right)^{\gamma} \tag{3.217}$$

$$\frac{T}{T_o} = \left(\frac{v_o}{v}\right)^{\gamma-1} = \left(\frac{P}{P_o}\right)^{\frac{\gamma-1}{\gamma}} \tag{3.218}$$

Find the work in a process from v_1 to v_2

$$w_{12} = \int_2^1 P dv \tag{3.219}$$

$$= P_o v_o^{\gamma} \int_{v_1}^{v_2} \frac{dv}{v^{\gamma}} \tag{3.220}$$

$$= P_o v_o^{\gamma} \left[\frac{v^{1-\gamma}}{1-\gamma}\right]_{v_1}^{v_2} \tag{3.221}$$

$$= \frac{P_o v_o^{\gamma}}{1-\gamma}\left(v_2^{1-\gamma} - v_1^{1-\gamma}\right) \tag{3.222}$$

$$= \frac{P_2 v_2 - P_1 v_1}{1-\gamma} \tag{3.223}$$

Also

$$de = \delta q - \delta w = 0 - \delta w \qquad \text{so} \tag{3.224}$$

$$e_2 - e_1 = \frac{P_2 v_2 - P_1 v_1}{\gamma - 1} \tag{3.225}$$

Figure 3.4 gives a sketch for the calorically perfect ideal gas undergoing an isentropic expansion in various planes.

Example 3.9

Virial Gas

Find the relationship between P and v for a virial gas with constant c_v which undergoes an isentropic process.

Virial Gas:

$$P = \frac{RT}{v-b} \tag{3.226}$$

This is van der Waals with $a = 0$ and c_v constant so:

$$e = c_v T + e_o \tag{3.227}$$

Thus

$$e = c_v \frac{P\,(v-b)}{R} + e_o \tag{3.228}$$

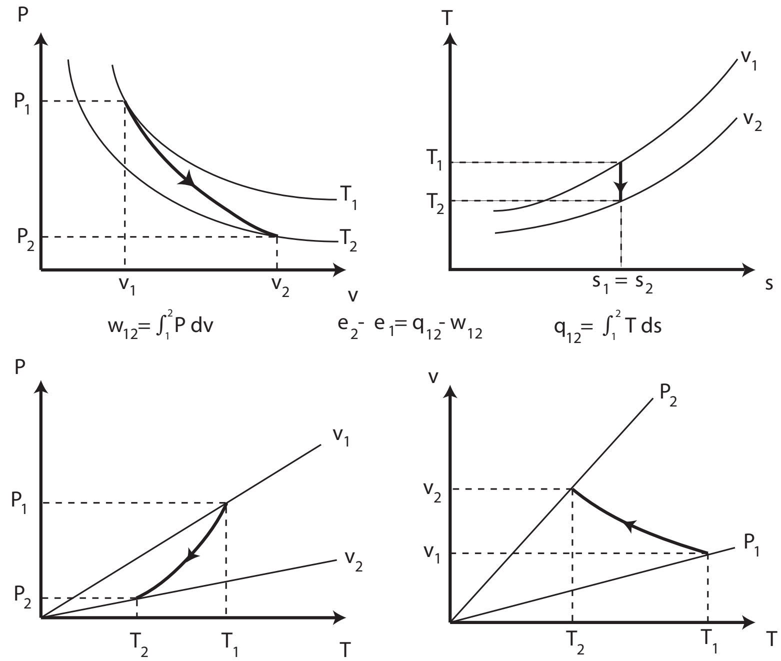

Figure 3.4: Sketch of isentropic expansion process

Thus the necessary derivatives are

$$\left.\frac{\partial e}{\partial P}\right|_v = \frac{c_v}{R}(v-b) \tag{3.229}$$

$$\left.\frac{\partial e}{\partial v}\right|_P = \frac{c_v}{R}P \tag{3.230}$$

so substituting into our developed relationship gives

$$\frac{c_v}{R}(v-b)\,dP + \left(\frac{c_v}{R}P + P\right)dv = 0 \tag{3.231}$$

$$(v-b)\,dP + \left(1+\frac{R}{c_v}\right)P dv = 0 \tag{3.232}$$

$$\text{with} \qquad \hat{\gamma} \equiv 1 + \frac{R}{c_v} \tag{3.233}$$

$$\frac{dP}{dv} + \frac{\hat{\gamma}}{v-b}P = 0 \tag{3.234}$$

$$\left(\exp\int\frac{\hat{\gamma}}{v-b}dv\right)\frac{dP}{dv} + \left(\exp\int\frac{\hat{\gamma}}{v-b}dv\right)\frac{\hat{\gamma}}{v-b}P = 0 \tag{3.235}$$

$$\left(\exp\left(\ln(v-b)^{\hat{\gamma}}\right)\right)\frac{dP}{dv} + \left(\exp\left(\ln(v-b)^{\hat{\gamma}}\right)\right)\frac{\hat{\gamma}}{v-b}P = 0 \tag{3.236}$$

$$(v-b)^{\hat{\gamma}}\frac{dP}{dv} + (v-b)^{\hat{\gamma}}\frac{\hat{\gamma}}{v-b}P = 0 \tag{3.237}$$

$$\frac{d}{dv}\left((v-b)^{\hat{\gamma}}P\right) = 0 \tag{3.238}$$

$$(v - b)^{\hat{\gamma}} P = (v_o - b)^{\hat{\gamma}} P_o \tag{3.239}$$

$$\frac{P}{P_o} = \left(\frac{v_o - b}{v - b}\right)^{\hat{\gamma}} \tag{3.240}$$

Exercise: Find the relationship between T and v for a virial gas in an isentropic process.

Exercise: Find an expression for the work done by a van der Waals gas in an isentropic process.

Exercise: A virial gas, $m = 3\ kg$ with $R = 290\ \frac{J}{kgK}$, $b = 0.002\ \frac{m^3}{kg}$ with constant specific heat $c_v = 0.700\ \frac{kJ}{kg\ K}$ is initially at $P = 1.2\ bar$ and $T = 320\ K$. It undergoes a two step process: $1 \rightarrow 2$ is an isochoric compression to $500\ kPa$; $2 \rightarrow 3$ is an isentropic expansion to $300\ kPa$. Find the total work W_{13} in units of J, the total heat transfer Q_{13} in units of J, and the change in entropy $S_3 - S_1$ in units of J/K. Include a sketch, roughly to scale, of the total process in the $P - v$ and $T - s$ planes.

Chapter 4

One-dimensional compressible flow

White, Chapter 9: pp. 511-559,
Liepmann and Roshko, Chapter 2: pp. 39-65,
Hughes and Brighton, Chapter 7: pp. 178-185,
Shapiro, Vol. 1, Chapters 4-8: pp. 73-262,

This chapter will discuss one-dimensional flow of a compressible fluid. Notation can pose problems, and many common ones are in use. Here a new convention will be adopted. In this chapter

- velocity in the x-direction will be denoted as u,
- specific internal energy, denoted in previous chapters by u, will here be e,
- total internal energy, denoted in previous chapters by U, will here be E.

The following topics will be covered:

- development of generalized one-dimensional flow equations,
- isentropic flow with area change,
- flow with normal shock waves,
- flow with friction (Fanno flow),
- flow with heat transfer (Rayleigh flow),
- flow in a shock tube.

Assume for this chapter:

- The flow is uni-directional in the $x-$ direction with $u \neq 0$ and with the $y-$ and $z-$ components of the velocity vector both zero: $v \equiv 0, w \equiv 0$

- Spatial gradients are admitted in x, but not in y or z: $\frac{\partial}{\partial x} \neq 0$, $\frac{\partial}{\partial y} \equiv 0$, $\frac{\partial}{\partial z} \equiv 0$.

Friction and heat transfer will not be modelled rigorously. Instead, they will be modelled in a fashion which captures the relevant physics and retains analytic tractability.

4.1 Generalized one-dimensional equations

Flow with area change is illustrated by the following sketch of a control volume:. See Figure 4.1.

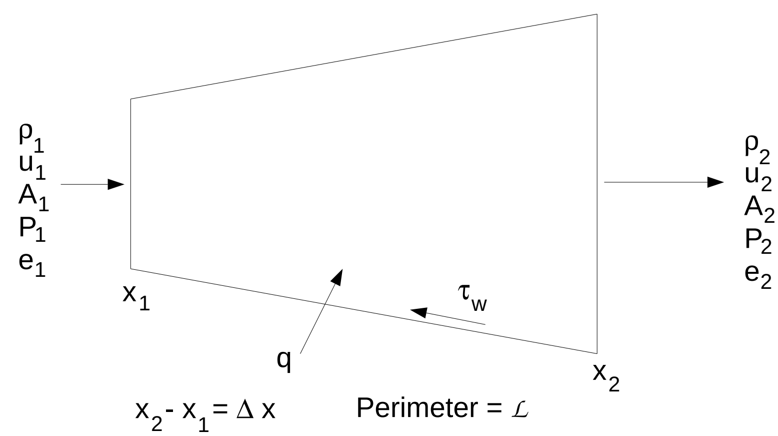

Figure 4.1: Control volume sketch

For this problem adopt the following conventions

- surface 1 and 2 are open and allow fluxes of mass, momentum, and energy
- surface w is a closed wall; no mass flux through the wall
- external heat flux q_w (Energy/Area/Time: $\frac{W}{m^2}$) *through* the wall allowed-q_w known *fixed* parameter
- diffusive, longitudinal heat transfer ignored, $q_x = 0$
- wall shear τ_w (Force/Area: $\frac{N}{m^2}$) allowed–τ_w known, *fixed* parameter
- diffusive viscous stress not allowed $\tau_{xx} = 0$
- cross-sectional area a known *fixed* function: $A(x)$

4.1.1 Mass

Take the overbar notation to indicate a volume averaged quantity.

The amount of mass in a control volume after a time increment Δt is equal to the original amount of mass plus that which came in minus that which left:

$$\bar{\rho}\bar{A}\Delta x\big|_{t+\Delta t} = \bar{\rho}\bar{A}\Delta x\big|_{t} + \rho_1 A_1 (u_1 \Delta t) - \rho_2 A_2 (u_2 \Delta t) \tag{4.1}$$

Rearrange and divide by $\Delta x \Delta t$:

$$\frac{\bar{\rho}\bar{A}\big|_{t+\Delta t} - \bar{\rho}\bar{A}\big|_{t}}{\Delta t} + \frac{\rho_2 A_2 u_2 - \rho_1 A_1 u_1}{\Delta x} = 0 \tag{4.2}$$

$$\tag{4.3}$$

Taking the limit as $\Delta t \to 0, \Delta x \to 0$:

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0 \tag{4.4}$$

If steady

$$\frac{d}{dx}(\rho A u) = 0 \tag{4.5}$$

$$A u \frac{d\rho}{dx} + \rho u \frac{dA}{dx} + \rho A \frac{du}{dx} = 0 \tag{4.6}$$

$$\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{A}\frac{dA}{dx} + \frac{1}{u}\frac{du}{dx} = 0 \tag{4.7}$$

Integrate from x_1 to x_2:

$$\int_{x_1}^{x_2} \frac{d}{dx}(\rho A u) dx = \int_{x_1}^{x_2} 0 dx \tag{4.8}$$

$$\int_1^2 d(\rho A u) = 0 \tag{4.9}$$

$$\rho_2 u_2 A_2 - \rho_1 u_1 A_1 = 0 \tag{4.10}$$

$$\rho_2 u_2 A_2 = \rho_1 u_1 A_1 \equiv \dot{m} = C_1 \tag{4.11}$$

4.1.2 Momentum

Newton's Second Law says the time rate of change of linear momentum of a body equals the sum of the forces acting on the body. In the x direction this is roughly as follows:

$$\frac{d}{dt}(mu) = F_x \tag{4.12}$$

In discrete form this would be

$$\frac{mu|_{t+\Delta t} - mu|_t}{\Delta t} = F_x \tag{4.13}$$

$$mu|_{t+\Delta t} = mu|_t + F_x \Delta t \tag{4.14}$$

For a control volume containing fluid, one must also account for the momentum which enters and leaves the control volume. The amount of momentum in a control volume after a time increment Δt is equal to the original amount of momentum plus that which came in minus that which left plus that introduced by the forces acting on the control volume.

- pressure force at surface 1 *pushes* fluid
- pressure force at surface 2 *restrains* fluid
- force due to the reaction of the wall to the pressure force *pushes* fluid if area change positive
- force due to the reaction of the wall to the shear force *restrains* fluid

$$\begin{aligned}\left(\bar{\rho}\bar{A}\Delta x\right)\bar{u}\Big|_{t+\Delta t} = \; & \left(\bar{\rho}\bar{A}\Delta x\right)\bar{u}\Big|_{t} \\ & + \left(\rho_1 A_1 \left(u_1 \Delta t\right)\right) u_1 \\ & - \left(\rho_2 A_2 \left(u_2 \Delta t\right)\right) u_2 \\ & + \left(P_1 A_1\right)\Delta t - \left(P_2 A_2\right)\Delta t \\ & + \left(\bar{P}\left(A_2 - A_1\right)\right)\Delta t \\ & - \left(\tau_w \bar{\mathcal{L}}\Delta x\right)\Delta t\end{aligned}$$

Rearrange and divide by $\Delta x \Delta t$:

$$\frac{\bar{\rho}\bar{A}\bar{u}\big|_{t+\Delta t} - \bar{\rho}\bar{A}\bar{u}\big|_{t}}{\Delta t} + \frac{\rho_2 A_2 u_2^2 - \rho_1 A_1 u_1^2}{\Delta x} = -\frac{P_2 A_2 - P_1 A_1}{\Delta x} + \bar{P}\frac{A_2 - A_1}{\Delta x} - \tau_w \bar{\mathcal{L}}$$

In the limit $\Delta x \to 0, \Delta t \to 0$ one gets

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}\left(\rho A u^2\right) = -\frac{\partial}{\partial x}(PA) + P\frac{\partial A}{\partial x} - \tau_w \mathcal{L} \tag{4.15}$$

In steady state:

$$\frac{d}{dx}\left(\rho A u^2\right) = -\frac{d}{dx}(PA) + P\frac{dA}{dx} - \tau_w \mathcal{L} \tag{4.16}$$

$$\begin{aligned}
\rho A u \frac{du}{dx} + u \frac{d}{dx}(\rho A u) &= -P\frac{dA}{dx} - A\frac{dP}{dx} + P\frac{dA}{dx} - \tau_w \mathcal{L} && (4.17)\\
\rho u \frac{du}{dx} &= -\frac{dP}{dx} - \tau_w \frac{\mathcal{L}}{A} && (4.18)\\
\rho u du + dP &= -\tau_w \frac{\mathcal{L}}{A} dx && (4.19)\\
du + \frac{1}{\rho u} dP &= -\tau_w \frac{\mathcal{L}}{\dot{m}} dx && (4.20)\\
\rho d\left(\frac{u^2}{2}\right) + dP &= -\tau_w \frac{\mathcal{L}}{A} dx && (4.21)
\end{aligned}$$

Wall shear lowers the combination of pressure and dynamic head.

If no wall shear:

$$dP = -\rho d\left(\frac{u^2}{2}\right) \tag{4.22}$$

Increase in velocity magnitude decreases the pressure.

If no area change $dA = 0$ and no friction $\tau_w \equiv 0$:

$$\begin{aligned}
\rho u \frac{du}{dx} + \frac{dP}{dx} &= 0 && (4.23)\\
\text{add } u \text{ mass} \qquad u\frac{d}{dx}(\rho u) &= 0 && (4.24)\\
\frac{d}{dx}\left(\rho u^2 + P\right) &= 0 && (4.25)\\
\rho u^2 + P &= \rho_o u_o^2 + P_o = C_2 && (4.26)
\end{aligned}$$

4.1.3 Energy

The first law of thermodynamics states that the change of total energy of a body equals the heat transferred to the body minus the work done by the body:

$$\begin{aligned}
E_2 - E_1 &= Q - W && (4.27)\\
E_2 &= E_1 + Q - W && (4.28)
\end{aligned}$$

So for the control volume this becomes the following when one also accounts for the energy flux in and out of the control volume in addition to the work and heat transfer:

$$\begin{aligned}
\left(\bar{\rho}\bar{A}\Delta x\right)\left(\bar{e} + \frac{\bar{u}^2}{2}\right)\Big|_{t+\Delta t} &= \left(\bar{\rho}\bar{A}\Delta x\right)\left(\bar{e} + \frac{\bar{u}^2}{2}\right)\Big|_{t}\\
&\quad + \rho_1 A_1 \left(u_1 \Delta t\right)\left(e_1 + \frac{u_1^2}{2}\right) - \rho_2 A_2 \left(u_2 \Delta t\right)\left(e_2 + \frac{u_2^2}{2}\right)\\
&\quad + q_w \left(\bar{\mathcal{L}}\Delta x\right)\Delta t + \left(P_1 A_1\right)\left(u_1 \Delta t\right) - \left(P_2 A_2\right)\left(u_2 \Delta t\right)
\end{aligned}$$

Note:

- mean pressure times area difference does no work because acting on stationary boundary
- work done by shear force not included[1]

Rearrange and divide by $\Delta t \Delta x$:

$$\frac{\bar{\rho}\bar{A}\left(\bar{e}+\frac{\bar{u}^2}{2}\right)\Big|_{t+\Delta t} - \bar{\rho}\bar{A}\left(\bar{e}+\frac{\bar{u}^2}{2}\right)\Big|_{t}}{\Delta t} + \frac{\rho_2 A_2 u_2\left(e_2+\frac{u_2^2}{2}+\frac{P_2}{\rho_2}\right) - \rho_1 A_1 u_1\left(e_1+\frac{u_1^2}{2}+\frac{P_1}{\rho_1}\right)}{\Delta x} = q_w \bar{\mathcal{L}}$$

In differential form as $\Delta x \to 0, \Delta t \to 0$

$$\frac{\partial}{\partial t}\left(\rho A\left(e+\frac{u^2}{2}\right)\right) + \frac{\partial}{\partial x}\left(\rho A u\left(e+\frac{u^2}{2}+\frac{P}{\rho}\right)\right) = q_w \mathcal{L}$$

In steady state:

$$\frac{d}{dx}\left(\rho A u\left(e+\frac{u^2}{2}+\frac{P}{\rho}\right)\right) = q_w \mathcal{L} \tag{4.29}$$

$$\rho A u \frac{d}{dx}\left(e+\frac{u^2}{2}+\frac{P}{\rho}\right) + \left(e+\frac{u^2}{2}+\frac{P}{\rho}\right)\frac{d}{dx}(\rho A u) = q_w \mathcal{L} \tag{4.30}$$

$$\rho u \frac{d}{dx}\left(e+\frac{u^2}{2}+\frac{P}{\rho}\right) = \frac{q_w \mathcal{L}}{A} \tag{4.31}$$

$$\rho u\left(\frac{de}{dx}+u\frac{du}{dx}+\frac{1}{\rho}\frac{dP}{dx}-\frac{P}{\rho^2}\frac{d\rho}{dx}\right) = \frac{q_w \mathcal{L}}{A} \tag{4.32}$$

subtract product of momentum and velocity (4.33)

$$\rho u^2 \frac{du}{dx} + u\frac{dP}{dx} = -\frac{\tau_w \mathcal{L} u}{A} \tag{4.34}$$

$$\rho u \frac{de}{dx} - \frac{Pu}{\rho}\frac{d\rho}{dx} = \frac{q_w \mathcal{L}}{A} + \frac{\tau_w \mathcal{L} u}{A} \tag{4.35}$$

$$\frac{de}{dx} - \frac{P}{\rho^2}\frac{d\rho}{dx} = \frac{\left(q_w + \tau_w u\right)\mathcal{L}}{\dot{m}} \tag{4.36}$$

[1] In neglecting work done by the wall shear force, I have taken an approach which is nearly universal, but fundamentally difficult to defend. At this stage of the development of these notes, I am not ready to enter into a grand battle with all established authors and probably confuse the student; consequently, results for flow with friction will be consistent with those of other sources. The argument typically used to justify this is that the real fluid satisfies no-slip at the boundary; thus, the wall shear actually does no work. However, one can easily argue that within the context of the one-dimensional model which has been posed that the shear force behaves as an external force which reduces the fluid's mechanical energy. Moreover, it is possible to show that neglect of this term results in the loss of frame invariance, a serious defect indeed. To model the work of the wall shear, one would include the term $\left(\tau_w\left(\bar{\mathcal{L}}\Delta x\right)\right)(\bar{u}\Delta t)$ in the energy equation.

Since $e = e(P, \rho)$

$$de = \left.\frac{\partial e}{\partial \rho}\right|_P d\rho + \left.\frac{\partial e}{\partial P}\right|_\rho dP \tag{4.37}$$

$$\frac{de}{dx} = \left.\frac{\partial e}{\partial \rho}\right|_P \frac{d\rho}{dx} + \left.\frac{\partial e}{\partial P}\right|_\rho \frac{dP}{dx} \tag{4.38}$$

so

$$\left.\frac{\partial e}{\partial \rho}\right|_P \frac{d\rho}{dx} + \left.\frac{\partial e}{\partial P}\right|_\rho \frac{dP}{dx} - \frac{P}{\rho^2}\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\dot{m}} \tag{4.39}$$

$$\frac{dP}{dx} + \left(\frac{\left.\frac{\partial e}{\partial \rho}\right|_P - \frac{P}{\rho^2}}{\left.\frac{\partial e}{\partial P}\right|_\rho}\right)\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\dot{m}\left.\frac{\partial e}{\partial P}\right|_\rho} \tag{4.40}$$

Now it can be shown that

$$c^2 = \left.\frac{\partial P}{\partial \rho}\right|_s = -\left(\frac{\left.\frac{\partial e}{\partial \rho}\right|_P - \frac{P}{\rho^2}}{\left.\frac{\partial e}{\partial P}\right|_\rho}\right) \tag{4.41}$$

so

$$\frac{dP}{dx} - c^2\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\dot{m}\left.\frac{\partial e}{\partial P}\right|_\rho} \tag{4.42}$$

$$\frac{dP}{dx} - c^2\frac{d\rho}{dx} = \frac{(q_w + \tau_w u)\,\mathcal{L}}{\rho u A\left.\frac{\partial e}{\partial P}\right|_\rho} \tag{4.43}$$

Special case of flow with no heat transfer $q_w \equiv 0$. Area change allowed!, wall friction allowed! (see earlier footnote):

$$\rho u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{P}{\rho}\right) = 0 \tag{4.44}$$

$$e + \frac{u^2}{2} + \frac{P}{\rho} = e_o + \frac{u_o^2}{2} + \frac{P_o}{\rho_o} = C_3 \tag{4.45}$$

$$h + \frac{u^2}{2} = h_o + \frac{u_o^2}{2} = C_3 \tag{4.46}$$

Example 4.1

Adiabatic Flow of Argon[2]

[2]adopted from White's 9.1, p. 583

Given: Argon, $\gamma = \frac{5}{3}$, flows adiabatically through a duct. At section 1, $P_1 = 200\ psia$, $T_1 = 500^\circ F, u_1 = 250\ \frac{ft}{s}$. At section 2 $P_2 = 40\ psia$, $u_2 = 1,100\ \frac{ft}{s}$.

Find: T_2 in $^\circ F$ and $s_2 - s_1$ in $\frac{Btu}{lbm^\circ R}$

Assume: Ar is a calorically perfect ideal gas, tables give $R = 38.68\ \frac{ft\ lbf}{lbmR}, c_p = 0.1253\ \frac{Btu}{lbmR}$

Analysis: First get the units into shape:

$$T_1 = 500 + 460 = 960\ R \quad (4.47)$$

$$c_p = \left(0.1253\ \frac{Btu}{lbm\ R}\right)\left(779 \frac{ft\ lbf}{Btu}\right)\left(32.17 \frac{lbm\ ft}{lbf\ s^2}\right) = 3,140\ \frac{ft^2}{s^2\ R} \quad (4.48)$$

$$R = \left(38.68\ \frac{ft\ lbf}{lbm\ R}\right)\left(32.17 \frac{lbm\ ft}{lbf\ s^2}\right) = 1,244\ \frac{ft^2}{s^2\ R} \quad (4.49)$$

$$R = \left(38.68\ \frac{ft\ lbf}{lbm\ R}\right)\left(\frac{1}{779}\frac{Btu}{ft\ lbf}\right) = 0.04965\ \frac{Btu}{lbm\ R} \quad (4.50)$$

Now consider an energy balance:

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{u_1^2}{2} \quad (4.51)$$

$$c_pT_2 + h_o + \frac{u_2^2}{2} = c_pT_1 + h_o + \frac{u_1^2}{2} \quad (4.52)$$

$$T_2 = T_1 + \frac{1}{2c_p}\left(u_1^2 - u_2^2\right) \quad (4.53)$$

$$T_2 = 960\ R + \frac{1}{2}\frac{1}{3,140\ \frac{ft^2}{s^2 R}}\left(\left(250\ \frac{ft}{s}\right)^2 - \left(1,100\ \frac{ft}{s}\right)^2\right) = 777\ R \quad (4.54)$$

$$T_2 = 777 - 460 = 317^\circ F \quad (4.55)$$

The flow sped up; temperature went down. Thermal energy was converted into kinetic energy

Calculate the entropy change. For the calorically perfect ideal gas:

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \quad (4.56)$$

$$= 0.1253\ \frac{Btu}{lbm\ R}\ln\left(\frac{777\ R}{960\ R}\right) - 0.04965\ \frac{Btu}{lbm\ R}\ln\left(\frac{40\ psia}{200\ psia}\right) \quad (4.57)$$

$$= -0.0265 - (-.0799) = 0.0534\ \frac{Btu}{lbm\ R} \quad (4.58)$$

Entropy change positive. Since adiabatic, there must have been irreversible friction which gave rise to this.

Example 4.2

Adiabatic Flow of Steam[3]

[3]adopted from White's 9.2, p. 583

Same problem now with steam Given: Steam flows adiabatically through a duct. At section 1, $P_1 = 200\ psia$, $T_1 = 500°F, u_1 = 250\ \frac{ft}{s}$. At section 2 $P_2 = 40\ psia$, $u_2 = 1,100\ \frac{ft}{s}$.

Find: T_2 in $°F$ and $s_2 - s_1$ in $\frac{Btu}{lbm°R}$

Analysis:
Use steam tables for property values.
Energy balance:

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{u_1^2}{2} \tag{4.59}$$

$$h_2 = h_1 + \frac{1}{2}\left(u_1^2 - u_2^2\right) \tag{4.60}$$

$$h_2 = 1269\ \frac{Btu}{lbm} + \frac{1}{2}\left(\frac{1}{779}\frac{Btu}{ft\ lbf}\right)\left(\frac{1}{32.17}\frac{lbf\ s^2}{lbm\ ft}\right)\left(\left(250\ \frac{ft}{s}\right)^2 - \left(1,100\ \frac{ft}{s}\right)^2\right) \tag{4.61}$$

$$h_2 = 1,246\ \frac{Btu}{lbm} \tag{4.62}$$

Interpolate steam tables at $P_2 = 40\ psia$, $h_2 = h_2 = 1,246\ \frac{Btu}{lbm}$ and find

$$T_2 = 420°F \tag{4.63}$$

$$s_2 = 1.7720\ \frac{Btu}{lbm\ R} \tag{4.64}$$

Tables give $s_1 = 1.6239\ \frac{Btu}{lbm\ R}$ so the entropy change is

$$s_2 - s_1 = 1.7720 - 1.6239 = 0.148\ \frac{Btu}{lbm\ R} \tag{4.65}$$

Example 4.3
Flow of Air with Heat Addition

Given: Air initially at $P_1 = 100\ kPa$, $T_1 = 300\ K$, $u_1 = 10\ \frac{m}{s}$ flows in a duct of length 100 m. The duct has a constant circular cross sectional area of $A = 0.02\ m^2$ and is isobarically heated with a constant heat flux q_w along the entire surface of the duct. At the end of the duct the flow has $P_2 = 100\ kPa$, $T_2 = 500\ K$

Find: the mass flow rate $\dot{m}$, the wall heat flux q_w and the entropy change $s_2 - s_1$; check for satisfaction of the second law.

Assume: Calorically perfect ideal gas, $R = 0.287\ \frac{kJ}{kg\ K}$, $c_p = 1.0035\ \frac{kJ}{kg\ K}$

Analysis:

Geometry:

$$A = \pi r^2 \tag{4.66}$$

$$r = \sqrt{\frac{A}{\pi}} \tag{4.67}$$

$$\mathcal{L} = 2\pi r = 2\sqrt{\pi A} = 2\sqrt{\pi\,(0.02\ m^2)} = 0.501\ m \tag{4.68}$$

Get the mass flux.

$$P_1 = \rho_1 R T_1 \tag{4.69}$$

$$\rho_1 = \frac{P_1}{RT_1} = \frac{100\ kPa}{\left(0.287\ \frac{kJ}{kg\ K}\right)(300\ K)} \tag{4.70}$$

$$= 1.161\ \frac{kg}{m^3} \tag{4.71}$$

So

$$\dot{m} = \rho_1 u_1 A_1 = \left(1.161\ \frac{kg}{m^3}\right)\left(10\ \frac{m}{s}\right)\left(0.02\ m^2\right) = 0.2322\ \frac{kg}{s} \tag{4.72}$$

Get the flow variables at state 2:

$$\rho_2 = \frac{P_2}{RT_2} = \frac{100\ kPa}{\left(0.287\ \frac{kJ}{kg\ K}\right)(500\ K)} \tag{4.73}$$

$$= 0.6969\ \frac{kg}{m^3} \tag{4.74}$$

$$\rho_2 u_2 A_2 = \rho_1 u_1 A_1 \tag{4.75}$$

$$u_2 = \frac{\rho_1 u_1 A_1}{\rho_2 A_2} = \frac{\rho_1 u_1}{\rho_2} \tag{4.76}$$

$$= \frac{\left(1.161\ \frac{kg}{m^3}\right)\left(10\ \frac{m}{s}\right)}{0.6969\ \frac{kg}{m^3}} = 16.67\ \frac{m}{s} \tag{4.77}$$

Now consider the energy equation:

$$\rho u \frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{P}{\rho}\right) = \frac{q_w \mathcal{L}}{A} \tag{4.78}$$

$$\frac{d}{dx}\left(h + \frac{u^2}{2}\right) = \frac{q_w \mathcal{L}}{\dot{m}} \tag{4.79}$$

$$\int_0^L \frac{d}{dx}\left(h + \frac{u^2}{2}\right)dx = \int_0^L \frac{q_w \mathcal{L}}{\dot{m}}dx \tag{4.80}$$

$$h_2 + \frac{u_2^2}{2} - h_1 - \frac{u_1^2}{2} = \frac{q_w L \mathcal{L}}{\dot{m}} \tag{4.81}$$

$$c_p\left(T_2 - T_1\right) + \frac{u_2^2}{2} - \frac{u_1^2}{2} = \frac{q_w L \mathcal{L}}{\dot{m}} \tag{4.82}$$

$$q_w = \left(\frac{\dot{m}}{L\mathcal{L}}\right)\left(c_p\left(T_2 - T_1\right) + \frac{u_2^2}{2} - \frac{u_1^2}{2}\right) \tag{4.83}$$

$$\tag{4.84}$$

Substituting the numbers, one finds,

$$q_w = \left(\frac{0.2322\ \frac{kg}{s}}{(100\ m)\,(0.501\ m)}\right)\left(1{,}003.5\frac{J}{kg\ K}\,(500\ K - 300\ K) + \frac{\left(16.67\ \frac{m}{s}\right)^2}{2} - \frac{\left(10\ \frac{m}{s}\right)^2}{2}\right) \quad (4.85)$$

$$q_w = 0.004635\ \frac{kg}{m^2\ s}\left(200{,}700\ \frac{J}{kg} - 88.9\ \frac{m^2}{s^2}\right) \quad (4.86)$$

$$q_w = 0.004635\ \frac{kg}{m^2\ s}\left(200{,}700\ \frac{J}{kg} - 88.9\ \frac{J}{kg}\right) \quad (4.87)$$

$$q_w = 930\ \frac{W}{m^2} \quad (4.88)$$

Heat flux positive, denoting heat into the air.

Now find the entropy change.

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \quad (4.89)$$

$$s_2 - s_1 = \left(1{,}003.5\ \frac{J}{kg\ K}\right)\ln\left(\frac{500\ K}{300\ K}\right) - \left(287\ \frac{J}{kg\ K}\right)\ln\left(\frac{100\ kPa}{100\ kPa}\right) \quad (4.90)$$

$$s_2 - s_1 = 512.6 - 0 = 512.6\ \frac{J}{kg\ K} \quad (4.91)$$

Is the second law satisfied? Assume the heat transfer takes place from a reservoir held at 500 K. The reservoir would have to be *at least* at 500 K in order to bring the fluid to its final state of 500 K. It could be greater than 500 K and still satisfy the second law.

$$S_2 - S_1 \geq \frac{Q_{12}}{T} \quad (4.92)$$

$$\dot{S}_2 - \dot{S}_1 \geq \frac{\dot{Q}_{12}}{T} \quad (4.93)$$

$$\dot{m}\,(s_2 - s_1) \geq \frac{\dot{Q}_{12}}{T} \quad (4.94)$$

$$\dot{m}\,(s_2 - s_1) \geq \frac{q_w A_{tot}}{T} \quad (4.95)$$

$$\dot{m}\,(s_2 - s_1) \geq \frac{q_w L\mathcal{L}}{T} \quad (4.96)$$

$$s_2 - s_1 \geq \frac{q_w L\mathcal{L}}{\dot{m}T} \quad (4.97)$$

$$512.6\ \frac{J}{kg\ K} \geq \frac{\left(930\ \frac{J}{s\ m^2}\right)(100\ m)\,(0.501\ m)}{\left(0.2322\ \frac{kg}{s}\right)(500\ K)} \quad (4.98)$$

$$512.6\ \frac{J}{kg\ K} \geq 401.3\ \frac{J}{kg\ K} \quad (4.99)$$

4.1.4 Influence coefficients

Now, uncouple these equations. First, summarize:

$$u\frac{d\rho}{dx}+\rho\frac{du}{dx} = -\frac{\rho u}{A}\frac{dA}{dx} \quad (4.100)$$

$$\rho u\frac{du}{dx}+\frac{dP}{dx} = -\frac{\tau_w\mathcal{L}}{A} \quad (4.101)$$

$$\frac{dP}{dx}-c^2\frac{d\rho}{dx} = \frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A\left.\frac{\partial e}{\partial P}\right|_\rho} \quad (4.102)$$

In matrix form this is

$$\begin{pmatrix} u & \rho & 0 \\ 0 & \rho u & 1 \\ -c^2 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{d\rho}{dx} \\ \frac{du}{dx} \\ \frac{dP}{dx} \end{pmatrix} = \begin{pmatrix} -\frac{\rho u}{A}\frac{dA}{dx} \\ -\frac{\tau_w\mathcal{L}}{A} \\ \frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A\left.\frac{\partial e}{\partial P}\right|_\rho} \end{pmatrix} \quad (4.103)$$

Use Cramer's Rule to solve for the derivatives. First calculate the determinant of the coefficient matrix:

$$u\left((\rho u)(1)-(1)(0)\right)-\rho\left((0)(1)-(-c^2)(1)\right)=\rho\left(u^2-c^2\right) \quad (4.104)$$

Implementing Cramer's Rule:

$$\frac{d\rho}{dx} = \frac{\rho u\left(-\frac{\rho u}{A}\frac{dA}{dx}\right)-\rho\left(-\frac{\tau_w\mathcal{L}}{A}\right)+\rho\left(\frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A\left.\frac{\partial e}{\partial P}\right|_\rho}\right)}{\rho\left(u^2-c^2\right)} \quad (4.105)$$

$$\frac{du}{dx} = \frac{-c^2\left(-\frac{\rho u}{A}\frac{dA}{dx}\right)+u\left(-\frac{\tau_w\mathcal{L}}{A}\right)-u\left(\frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A\left.\frac{\partial e}{\partial P}\right|_\rho}\right)}{\rho\left(u^2-c^2\right)} \quad (4.106)$$

$$\frac{dP}{dx} = \frac{\rho u c^2\left(-\frac{\rho u}{A}\frac{dA}{dx}\right)-\rho c^2\left(-\frac{\tau_w\mathcal{L}}{A}\right)+\rho u^2\left(\frac{(q_w+\tau_w u)\mathcal{L}}{\rho u A\left.\frac{\partial e}{\partial P}\right|_\rho}\right)}{\rho\left(u^2-c^2\right)} \quad (4.107)$$

Simplify

$$\frac{d\rho}{dx} = \frac{1}{A}\frac{-\rho u^2\frac{dA}{dx}+\tau_w\mathcal{L}+\frac{(q_w+\tau_w u)\mathcal{L}}{\rho u\left.\frac{\partial e}{\partial P}\right|_\rho}}{\left(u^2-c^2\right)} \quad (4.108)$$

$$\frac{du}{dx} = \frac{1}{A}\frac{c^2\rho u\frac{dA}{dx}-u\tau_w\mathcal{L}-\frac{(q_w+\tau_w u)\mathcal{L}}{\rho\left.\frac{\partial e}{\partial P}\right|_\rho}}{\rho\left(u^2-c^2\right)} \quad (4.109)$$

$$\frac{dP}{dx} = \frac{1}{A}\frac{-c^2\rho u^2\frac{dA}{dx}+c^2\tau_w\mathcal{L}+\frac{(q_w+\tau_w u)\mathcal{L}u}{\rho\left.\frac{\partial e}{\partial P}\right|_\rho}}{\left(u^2-c^2\right)} \quad (4.110)$$

Note:

- a system of coupled non-linear ordinary differential equations
- in standard form for dynamic system analysis: $\frac{d\mathbf{u}}{dx} = \mathbf{f}(\mathbf{u})$
- valid for *general* equations of state
- *singular* when velocity sonic $u = c$

4.2 Flow with area change

This section will consider flow with area change with an emphasis on isentropic flow. Some problems will involve non-isentropic flow but a detailed discussion of such flows will be delayed.

4.2.1 Isentropic Mach number relations

Take special case of

- $\tau_w = 0$
- $q_w = 0$
- calorically perfect ideal gas (CPIG)

Then

$$\frac{d}{dx}(\rho u A) = 0 \tag{4.111}$$

$$\frac{d}{dx}\left(\rho u^2 + P\right) = 0 \tag{4.112}$$

$$\frac{d}{dx}\left(e + \frac{u^2}{2} + \frac{P}{\rho}\right) = 0 \tag{4.113}$$

Integrate the energy equation with $h = e + P/\rho$

$$h + \frac{u^2}{2} = h_o + \frac{u_o^2}{2} \tag{4.114}$$

If one defines the "o" condition to be a condition of rest, then $u_o \equiv 0$. This is a **stagnation** condition. So

$$h + \frac{u^2}{2} = h_o \tag{4.115}$$

$$(h - h_o) + \frac{u^2}{2} = 0 \tag{4.116}$$

Since CPIG,

$$c_p(T - T_o) + \frac{u^2}{2} = 0 \tag{4.117}$$

$$T - T_o + \frac{u^2}{2c_p} = 0 \tag{4.118}$$

$$1 - \frac{T_o}{T} + \frac{u^2}{2c_pT} = 0 \tag{4.119}$$

Now note that

$$c_p = c_p\frac{c_p - c_v}{c_p - c_v} = \frac{c_p}{c_v}\frac{c_p - c_v}{\frac{c_p}{c_v} - 1} = \frac{\gamma R}{\gamma - 1} \tag{4.120}$$

so

$$1 - \frac{T_o}{T} + \frac{\gamma - 1}{2}\frac{u^2}{\gamma RT} = 0 \tag{4.121}$$

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}\frac{u^2}{\gamma RT} \tag{4.122}$$

Recall the sound speed and Mach number for a CPIG:

$$c^2 = \gamma RT \qquad \text{if} \qquad P = \rho RT, \qquad e = c_vT + e_o \tag{4.123}$$

$$M^2 \equiv \left(\frac{u}{c}\right)^2 \tag{4.124}$$

thus,

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}M^2 \tag{4.125}$$

$$\frac{T}{T_o} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{-1} \tag{4.126}$$

Now if the flow is isentropic one has

$$\frac{T}{T_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma - 1} = \left(\frac{P}{P_o}\right)^{\frac{\gamma - 1}{\gamma}} \tag{4.127}$$

Thus

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{1}{\gamma - 1}} \tag{4.128}$$

$$\frac{P}{P_o} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\frac{\gamma}{\gamma - 1}} \tag{4.129}$$

For air $\gamma = 7/5$ so

$$\frac{T}{T_o} = \left(1 + \frac{1}{5}M^2\right)^{-1} \tag{4.130}$$

$$\frac{\rho}{\rho_o} = \left(1 + \frac{1}{5}M^2\right)^{-\frac{5}{2}} \tag{4.131}$$

$$\frac{P}{P_o} = \left(1 + \frac{1}{5}M^2\right)^{-\frac{7}{2}} \tag{4.132}$$

Figures 4.2, 4.3 4.4 show the variation of T, ρ and P with M^2 for isentropic flow.

Other thermodynamic properties can be determined from these, e.g. sound speed:

$$\frac{c}{c_o} = \sqrt{\frac{\gamma RT}{\gamma RT_o}} = \sqrt{\frac{T}{T_o}} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{-1/2} \tag{4.133}$$

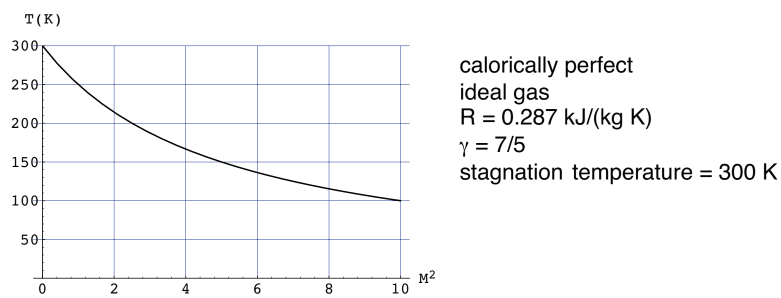

Figure 4.2: Static temperature versus Mach number squared

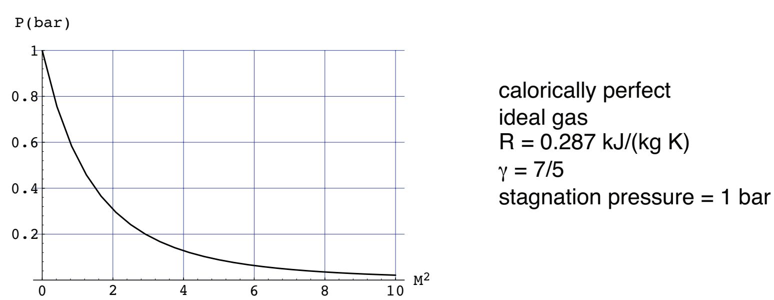

Figure 4.3: Static pressure versus Mach number squared

Example 4.4
Airplane problem

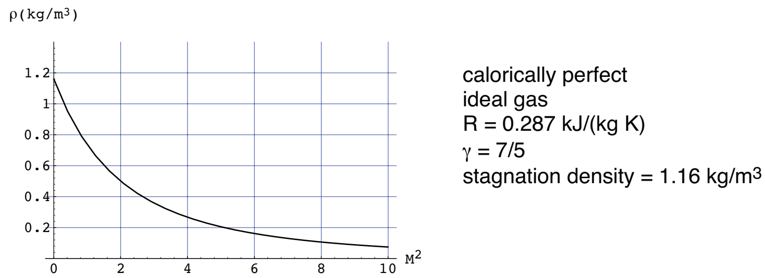

Figure 4.4: Static density versus Mach number squared

Given: An airplane is flying into still air at $u = 200\ m/s$. The ambient air is at 288 K and 101.3 kPa.

Find: Temperature, pressure, and density at nose of airplane

Assume: Steady isentropic flow of CPIG

Analysis: In the steady wave frame, the ambient conditions are *static* while the nose conditions are *stagnation*.

$$M = \frac{u}{c} = \frac{u}{\sqrt{\gamma RT}} = \frac{200\ m/s}{\sqrt{\frac{7}{5}\left(287\frac{J}{kgK}\right) 288\ K}} = 0.588 \tag{4.134}$$

so

$$T_o = T\left(1 + \frac{1}{5}M^2\right), \tag{4.135}$$

$$= (288\ K)\left(1 + \frac{1}{5}0.588^2\right), \tag{4.136}$$

$$= 307.9\ K \tag{4.137}$$

$$\rho_o = \rho\left(1 + \frac{1}{5}M^2\right)^{\frac{5}{2}} \tag{4.138}$$

$$= \frac{101.3\ kPa}{0.287\frac{kJ}{kgK}288\ K}\left(1 + \frac{1}{5}0.588^2\right)^{\frac{5}{2}}, \tag{4.139}$$

$$= 1.45\ kg/m^3 \tag{4.140}$$

$$P_o = P\left(1 + \frac{1}{5}M^2\right)^{\frac{7}{2}}, \tag{4.141}$$

$$= (101.3\ kPa)\left(1 + \frac{1}{5}0.588^2\right)^{\frac{7}{2}} \tag{4.142}$$

$$= 128\ kPa \tag{4.143}$$

Note the temperature, pressure, and density all rise in the isentropic process. In this wave frame, the kinetic energy of the flow is being converted isentropically to thermal energy.

Example 4.5

Pressure measurement in compressible flows[4]

See Figure 4.5.

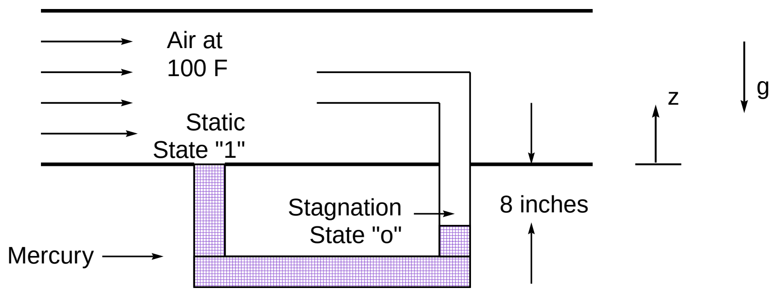

Figure 4.5: Compressible pitot tube sketch

Given: Air at $u = 750\ \frac{ft}{s}$, Mercury manometer which reads a change in height of 8 *inches*.

Find: Static pressure of air in *psia*

Assume: Ideal gas behavior for air

Analysis:

First consider the manometer which is governed by *fluid statics*. In fluid statics, there is no motion, thus there are no viscous forces or fluid inertia; one thus has a balance between surface and body forces. Consider the linear momentum equation:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P + \rho_{Hg}\mathbf{g} + \nabla \cdot \boldsymbol{\tau} \quad (4.144)$$

$$0 = -\nabla P + \rho_{Hg}\mathbf{g} \quad (4.145)$$

$$\frac{dP}{dz} = \rho_{Hg} g_z \quad (4.146)$$

$$P_1 - P_o = \rho_{Hg} g_z (z_1 - z_o) \quad (4.147)$$

$$P_1 - P_o = \left(845.9\ \frac{lbm}{ft^3}\right)\left(\frac{1}{32.2}\ \frac{lbf\ s^2}{ft\ lbm}\right)\left(-32.2\frac{ft}{s^2}\right)(0\ in - (-8\ in))\left(\frac{1}{12}\frac{ft}{in}\right) \quad (4.148)$$

$$P_1 - P_o = -563.9\ \frac{lbf}{ft^2} \quad (4.149)$$

$$P_o - P_1 = 563.9\ \frac{lbf}{ft^2} \quad (4.150)$$

$$P_o = P_1 + 563.9\ \frac{lbf}{ft^2} \quad (4.151)$$

[4]adopted from White, 9.26, p. 584

Now calculate the local Mach number

$$T_1 = 100 + 460 = 560\ R \quad (4.152)$$

$$M_1 = \frac{u_1}{\sqrt{\gamma R T_1}} \quad (4.153)$$

$$M_1 = \frac{750\ \frac{ft}{s}}{\sqrt{(1.4)\left(1,717\ \frac{ft^2}{s^2\ R}\right)(560\ R)}} \quad (4.154)$$

$$M_1 = 0.646 \quad (4.155)$$

Isentropic flow relations relate stagnation to static properties, so *for air*

$$P_o = P_1\left(1 + \frac{1}{5}M_1^2\right)^{3.5} \quad (4.156)$$

$$P_o = P_1\left(1 + \frac{1}{5}(0.646)^2\right)^{3.5} \quad (4.157)$$

$$P_o = 1.324 P_1 \quad (4.158)$$

Substituting from the measured pressure difference

$$P_1 + 563.9\ \frac{lbf}{ft^2} = 1.324 P_1 \quad (4.159)$$

$$-0.324 P_1 = -563.9\ \frac{lbf}{ft^2} \quad (4.160)$$

$$P_1 = \frac{-563.9\ \frac{lbf}{ft^2}}{-0.324} \quad (4.161)$$

$$P_1 = 1,740\ \frac{lbf}{ft^2} \quad (4.162)$$

$$P_o = (1.324)\left(1,740\ \frac{lbf}{ft^2}\right) = 2,304\ \frac{lbf}{ft^2} \quad (4.163)$$

$$P_1 = \left(1,740\ \frac{lbf}{ft^2}\right)\left(\frac{1}{12}\frac{ft}{in}\right)^2 = 12.1\ psia \quad (4.164)$$

$$P_o = \left(2,304\ \frac{lbf}{ft^2}\right)\left(\frac{1}{12}\frac{ft}{in}\right)^2 = 16.0\ psia \quad (4.165)$$

What might one estimate if one did not account for compressibility effects? Assume one had the same static pressure and calculate what velocity one would predict.

First calculate the static density.

$$\rho_1 = \frac{P_1}{RT_1} \quad (4.166)$$

$$\rho_1 = \frac{1,740\ \frac{lbf}{ft^2}}{\left(1,717\ \frac{ft^2}{s^2\ R}\right)(560\ R)}\left(32.2\ \frac{ft\ lbm}{lbf\ s^2}\right) \quad (4.167)$$

$$\rho_1 = 0.05827\ \frac{lbm}{ft^3} \quad (4.168)$$

One would then use an incompressible Bernoulli equation:

$$P_o + \frac{\rho\,(0)^2}{2} = P_1 + \frac{\rho_1 u_1^2}{2} \tag{4.169}$$

$$u_1 = \sqrt{\frac{2\,(P_o - P_1)}{\rho_1}} \tag{4.170}$$

$$u_1 = \sqrt{\frac{2\left(563.9\ \frac{lbf}{ft^2}\right)}{0.05827\ \frac{lbm}{ft^3}}\left(32.2\ \frac{ft\ lbm}{lbf\ s^2}\right)} \tag{4.171}$$

$$u_1 = 789.4\ \frac{ft}{s} \tag{4.172}$$

So the relative error in using the incompressible approximation would be

$$Error = \frac{789.4 - 750}{750} = 5.3\% \tag{4.173}$$

Example 4.6

Adiabatic Duct Flow[5]

Given: Air flowing adiabatically through a duct. At section 1, $u_1 = 400\ \frac{ft}{s}, T_1 = 200°F, P_1 = 35\ psia$. Downstream $u_2 = 1,100\ \frac{ft}{s}, P_2 = 18\ psia$.

Find: $M_2, u_{max}, \frac{P_{o2}}{P_{o1}}$

Assume: Calorically perfect ideal gas, steady, one-dimensional flow

Analysis:

Some preliminaries:

$$T_1 = 200 + 460 = 660\ R \tag{4.174}$$

$$c_p = \left(0.240\ \frac{Btu}{lbm\ R}\right)\left(779\frac{ft\ lbf}{Btu}\right)\left(32.17\frac{lbm\ ft}{lbf\ s^2}\right) = 6,015\ \frac{ft^2}{s^2\ R} \tag{4.175}$$

$$R = \left(53.34\ \frac{ft\ lbf}{lbm\ R}\right)\left(32.17\frac{lbm\ ft}{lbf\ s^2}\right) = 1,716\ \frac{ft^2}{s^2\ R} \tag{4.176}$$

(4.177)

Energy conservation gives stagnation conditions at state 1

$$h_1 + \frac{u_1^2}{2} = h_{o1} \tag{4.178}$$

[5]adopted from White's 9.30, p. 585

$$c_pT_1 + \frac{u_1^2}{2} = c_pT_{o1} \tag{4.179}$$

$$T_{o1} = T_1 + \frac{u_1^2}{2c_p} \tag{4.180}$$

$$T_{o1} = 660\ R + \frac{\left(400\ \frac{ft}{s}\right)^2}{2\left(6,015\ \frac{ft^2}{s^2\ R}\right)} \tag{4.181}$$

$$T_{o1} = 673\ R \tag{4.182}$$

Note since in adiabatic flow h_o is a constant, $h_{o2} = h_{o1}$ and since ideal gas $T_{o2} = T_{o1}$ So

$$T_{o2} = 673\ R \tag{4.183}$$

$$T_2 = T_{o2} - \frac{u_2^2}{2c_p} \tag{4.184}$$

$$T_2 = 673\ R - \frac{\left(1,100\ \frac{ft}{s}\right)^2}{2\left(6,015\ \frac{ft^2}{s^2\ R}\right)} \tag{4.185}$$

$$T_2 = 572\ R \tag{4.186}$$

Calculate the Mach numbers:

$$c_1 = \sqrt{\gamma R T_1} \tag{4.187}$$

$$c_1 = \sqrt{1.4\left(1,716\ \frac{ft^2}{s^2\ R}\right)(660\ R)} = 1,259\ \frac{ft}{s} \tag{4.188}$$

$$M_1 = \frac{u_1}{c_1} = \frac{400\ \frac{ft}{s}}{1,259\ \frac{ft}{s}} = 0.318 \tag{4.189}$$

$$c_2 = \sqrt{\gamma R T_2} \tag{4.190}$$

$$c_2 = \sqrt{1.4\left(1,716\ \frac{ft^2}{s^2\ R}\right)(572\ R)} = 1,173\ \frac{ft}{s} \tag{4.191}$$

$$M_2 = \frac{u_2}{c_2} = \frac{1,100\ \frac{ft}{s}}{1,173\ \frac{ft}{s}} = 0.938 \tag{4.192}$$

Since for CPIG air one has

$$\frac{P}{P_o} = \left(1 + \frac{1}{5}M^2\right)^{-\frac{7}{2}} \tag{4.193}$$

$$P_o = P\left(1 + \frac{1}{5}M^2\right)^{\frac{7}{2}} \tag{4.194}$$

$$P_{o1} = (35\ psia)\left(1 + \frac{1}{5}0.318^2\right)^{\frac{7}{2}} = 37.54\ psia \tag{4.195}$$

$$P_{o2} = (18\ psia)\left(1 + \frac{1}{5}0.938^2\right)^{\frac{7}{2}} = 31.74\ psia \tag{4.196}$$

$$\frac{P_{o2}}{P_{o1}} = \frac{31.74\ psia}{37.54\ psia} = 0.845 \tag{4.197}$$

Stagnation pressure drop indicates that friction was present. If one computed an entropy change one would see an increase in entropy.

The maximum velocity is found by converting all the thermal energy to kinetic energy. Taking zero thermal energy to correspond to absolute zero (despite the fact that air would not be a gas at this point) one could estimate

$$h_o = \frac{u_{max}^2}{2} \quad (4.198)$$

$$c_p T_o = \frac{u_{max}^2}{2} \quad (4.199)$$

$$u_{max} = \sqrt{2 c_p T_o} \quad (4.200)$$

$$u_{max} = \sqrt{2\left(6,015\ \frac{ft^2}{s^2\ R}\right)(673\ R)} = 2,845\ \frac{ft}{s} \quad (4.201)$$

4.2.2 Sonic properties

Let "*" denote a property at the sonic state $M^2 \equiv 1$

$$\frac{T_*}{T_o} = \left(1 + \frac{\gamma - 1}{2} 1^2\right)^{-1} = \frac{2}{\gamma + 1} \quad (4.202)$$

$$\frac{\rho_*}{\rho_o} = \left(1 + \frac{\gamma - 1}{2} 1^2\right)^{-\frac{1}{\gamma - 1}} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}} \quad (4.203)$$

$$\frac{P_*}{P_o} = \left(1 + \frac{\gamma - 1}{2} 1^2\right)^{-\frac{\gamma}{\gamma - 1}} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} \quad (4.204)$$

$$\frac{c_*}{c_o} = \left(1 + \frac{\gamma - 1}{2} 1^2\right)^{-1/2} = \sqrt{\frac{2}{\gamma + 1}} \quad (4.205)$$

$$u_* = c_* = \sqrt{\gamma R T_*} = \sqrt{\frac{2\gamma}{\gamma + 1} R T_o} \quad (4.206)$$

If air $\gamma = 7/5$ and

$$\frac{T_*}{T_o} = 0.8333 \quad (4.207)$$

$$\frac{\rho_*}{\rho_o} = 0.6339 \quad (4.208)$$

$$\frac{P_*}{P_o} = 0.5283 \quad (4.209)$$

$$\frac{c_*}{c_o} = 0.9123 \quad (4.210)$$

4.2.3 Effect of area change

Influence of mass equation must be considered. So far only looked at energy has been examined. In the isentropic limit the mass, momentum, and energy equation for a CPIG reduce to

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{4.211}$$

$$\rho u du + dP = 0 \tag{4.212}$$

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho} \tag{4.213}$$

Substitute energy then mass into momentum:

$$\rho u du + \gamma \frac{P}{\rho} d\rho = 0 \tag{4.214}$$

$$\rho u du + \gamma \frac{P}{\rho} \left(-\frac{\rho}{u} du - \frac{\rho}{A} dA \right) = 0 \tag{4.215}$$

$$du + \gamma \frac{P}{\rho} \left(-\frac{1}{u^2} du - \frac{1}{uA} dA \right) = 0 \tag{4.216}$$

$$du \left(1 - \frac{\gamma P/\rho}{u^2} \right) = \gamma \frac{P}{\rho} \frac{dA}{uA} \tag{4.217}$$

$$\frac{du}{u} \left(1 - \frac{\gamma P/\rho}{u^2} \right) = \frac{\gamma P/\rho}{u^2} \frac{dA}{A} \tag{4.218}$$

$$\frac{du}{u} \left(1 - \frac{1}{M^2} \right) = \frac{1}{M^2} \frac{dA}{A} \tag{4.219}$$

$$\frac{du}{u} \left(M^2 - 1 \right) = \frac{dA}{A} \tag{4.220}$$

$$\frac{du}{u} = \frac{1}{M^2 - 1} \frac{dA}{A} \tag{4.221}$$

Figure 4.6 gives show the performance of a fluid in a variable area duct.
It is noted that

- equation singular when $M^2 = 1$
- if $M^2 = 1$, one needs $dA = 0$
- area minimum necessary to transition from subsonic to supersonic flow!!
- can be shown area maximum not relevant

Consider A at a sonic state. From the mass equation:

$$\rho u A = \rho_* u_* A_* \tag{4.222}$$

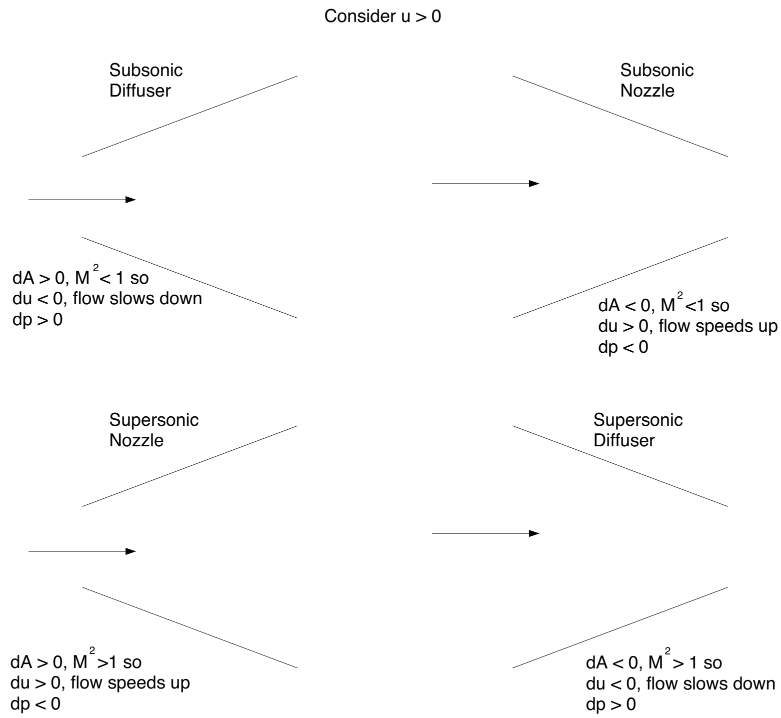

Figure 4.6: Behavior of fluid in sub- and supersonic nozzles and diffusers

$$\rho u A = \rho_* c_* A_* \tag{4.223}$$

$$\frac{A}{A_*} = \frac{\rho_*}{\rho} c_* \frac{1}{u}, \tag{4.224}$$

$$= \frac{\rho_*}{\rho} \sqrt{\gamma R T_*} \frac{1}{u}, \tag{4.225}$$

$$= \frac{\rho_*}{\rho} \frac{\sqrt{\gamma R T_*}}{\sqrt{\gamma R T}} \frac{\sqrt{\gamma R T}}{u} \tag{4.226}$$

$$= \frac{\rho_*}{\rho} \sqrt{\frac{T_*}{T}} \frac{1}{M}, \tag{4.227}$$

$$= \frac{\rho_*}{\rho_o} \frac{\rho_o}{\rho} \sqrt{\frac{T_* T_o}{T_o T}} \frac{1}{M} \tag{4.228}$$

Substitute from earlier-developed relations and get:

$$\frac{A}{A_*} = \frac{1}{M} \left(\frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \right)^{\frac{1}{2} \frac{\gamma+1}{\gamma-1}} \tag{4.229}$$

Figure 4.7 shows the performance of a fluid in a variable area duct.

Note:

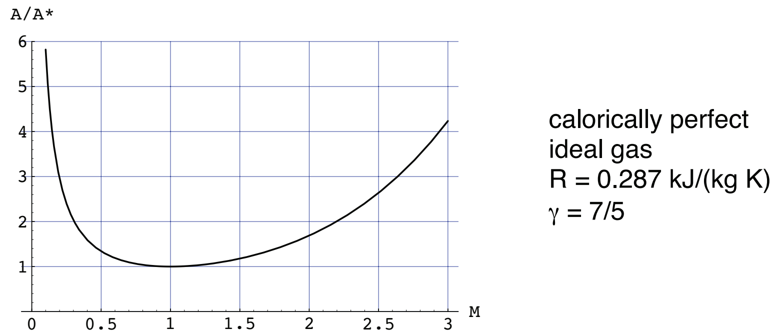

Figure 4.7: Area versus Mach number for a calorically perfect ideal gas

- $\frac{A}{A_*}$ has a minimum value of 1 at $M = 1$
- For each $\frac{A}{A*} > 1$, there exist **two** values of M
- $\frac{A}{A_*} \to \infty$ as $M \to 0$ or $M \to \infty$

4.2.4 Choking

Consider mass flow rate variation with pressure difference

- small pressure difference gives small velocity, small mass flow
- as pressure difference grows, velocity and mass flow rate grow
- velocity is limited to sonic at a particular duct location
- this provides fundamental restriction on mass flow rate
- can be proven rigorously that sonic condition gives maximum mass flow rate

$$\dot{m}_{max} = \rho_* u_* A_* \tag{4.230}$$

$$\textit{if}\text{ ideal gas:} \quad = \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\sqrt{\frac{2\gamma}{\gamma+1} R T_o}\right) A_* \tag{4.231}$$

$$= \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \left(\frac{2}{\gamma+1}\right)^{1/2} \sqrt{\gamma R T_o} A_* \tag{4.232}$$

$$= \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o} A_* \tag{4.233}$$

A flow which has a maximum mass flow rate is known as **choked** flow. Flows will choke at area minima in a duct.

Example 4.7

Isentropic area change problem with choking[6]

Given: Air with stagnation conditions $P_o = 200\ kPa$ $T_o = 500\ K$ flows through a throat to an exit Mach number of 2.5. The desired mass flow is 3.0 kg/s,

Find: a) throat area, b) exit pressure, c) exit temperature, d) exit velocity, and e) exit area.

Assume: CPIG, isentropic flow, $\gamma = 7/5$

Analysis:

$$\rho_o = \frac{P_o}{RT_o} = \frac{200\ kPa}{(0.287\ kJ/kg)\,(500\ K)} = 1.394\ kg/m^3 \tag{4.234}$$

Since it necessarily flows through a sonic throat:

$$\dot{m}_{max} = \rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o} A_* \tag{4.235}$$

$$A_* = \frac{\dot{m}_{max}}{\rho_o \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o}} \tag{4.236}$$

$$= \frac{3\ kg/s}{\left(1.394\ \frac{kg}{m^3}\right)(0.5787)\sqrt{1.4\left(287\ \frac{J}{kg\ K}\right)(500\ K)}}, \tag{4.237}$$

$$= 0.008297\ m^2 \tag{4.238}$$

Since M_e is known, use the isentropic relations to find other exit conditions.

$$P_e = P_o\left(1 + \frac{\gamma-1}{2}M_e^2\right)^{-\frac{\gamma}{\gamma-1}}, \tag{4.239}$$

$$= (200\ kPa)\left(1 + \frac{1}{5}2.5^2\right)^{-3.5}, \tag{4.240}$$

$$= 11.71\ kPa \tag{4.241}$$

$$T_e = T_o\left(1 + \frac{\gamma-1}{2}M_e^2\right)^{-1}, \tag{4.242}$$

$$= (500\ K)\left(1 + \frac{1}{5}2.5^2\right)^{-1}, \tag{4.243}$$

$$= 222.2\ K \tag{4.244}$$

Note

$$\rho_e = \frac{P_e}{RT_e}, \tag{4.245}$$

$$= \frac{11.71\ kPa}{\left(0.287\ \frac{kJ}{kgK}\right)(222.2\ K)}, \tag{4.246}$$

$$= 0.1834\ \frac{kg}{m^3} \tag{4.247}$$

[6]adopted from White, *Fluid Mechanics* McGraw-Hill: New York, 1986, p. 529, Ex. 9.5

Now the exit velocity is simply

$$u_e = M_e c_e = M_e \sqrt{\gamma R T_e} = 2.5 \sqrt{1.4 \left(287 \, \frac{J}{kg \; K}\right)(222.2 \; K)} = 747.0 \, \frac{m}{s} \tag{4.248}$$

Now determine the exit area:

$$A = \frac{A_*}{M_e}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_e^2\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \tag{4.249}$$

$$= \frac{0.008297 \; m^2}{2.5}\left(\frac{5}{6}\left(1+\frac{1}{5}2.5^2\right)\right)^3, \tag{4.250}$$

$$= 70.0219 \; m^2 \tag{4.251}$$

Example 4.8

Discharge Problem[7]

Given: Air in tank, $P_o = 700 \; kPa$, $T_o = 20°C$, $V = 1.5 \; m^3$. Throat area in converging nozzle of $0.65 \; cm^2$, exhausting to 1 atm environment

Find: Time for pressure in tank to decrease to 500 kPa.

Assume: CPIG, stagnation temperature constant (so small heat transfer to tank in time of operation)

Analysis:
First, $T_o = 20 + 273 = 293 \; K$

Now check for choked flow! At the initial state

$$\frac{P_{atm}}{P_o} = \frac{101.3 \; kPa}{700 \; kPa} = 0.145 \tag{4.252}$$

But for air $\frac{P_*}{P_o} = 0.5283$, so the flow must be **choked** at the exit and the mass flow is restricted. (Further expansion takes place outside the nozzle)

For choked flow one has

$$\dot{m}_e = \rho_o\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}\sqrt{\gamma R T_o} A_* \tag{4.253}$$

$$= \left(\frac{P_o}{RT_o}\right)\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}\sqrt{\gamma R T_o} A_* \tag{4.254}$$

$$= \left(\frac{P_o}{\left(287 \, \frac{J}{kg \; K}\right)(293 \; K)}\right) 0.5787 \sqrt{1.4\left(287 \, \frac{J}{kg \; K}\right)(293 \; K)}\left(0.65 \; cm^2\right)\left(\frac{1}{100}\frac{m}{cm}\right)^2 \tag{4.255}$$

$$= 1.5348 \times 10^{-7} P_o \tag{4.256}$$

[7] from White, 9.33,35

Now mass conservation gives

$$\frac{d}{dt}m_{cv} = -\dot{m}_e \quad (4.257)$$

$$\frac{d}{dt}(\rho_o V) = -\dot{m}_e \quad (4.258)$$

$$\frac{d}{dt}\left(\frac{P_o}{RT_o}V\right) = -\dot{m}_e \quad (4.259)$$

$$\frac{dP_o}{dt} = -\frac{RT_o}{V}\dot{m}_e \quad (4.260)$$

$$\frac{dP_o}{dt} = -\frac{\left(287\ \frac{J}{kg\ K}\right)(293\ K)}{1.5\ m^3}1.5348\times 10^{-7}P_o \quad (4.261)$$

$$\frac{dP_o}{dt} = -0.008604 P_o \quad (4.262)$$

$$P_o = A\exp(-0.008604t) \quad (4.263)$$

Use initial value of P_o to fix the constant A so

$$P_o = 700\exp(-0.008604t) \quad (4.264)$$

When does $P_o = 500\ kPa$?

$$500 = 700\exp(-0.008604t)\ln\frac{500}{700} = -0.008604t \quad (4.265)$$

$$t = -\frac{1}{0.008604}\ln\frac{500}{700} = 39.1\ s \quad (4.266)$$

4.3 Normal shock waves

This section will develop relations for normal shock waves in fluids with general equations of state. It will be specialized to calorically perfect ideal gases to illustrate the general features of the waves.

Assumptions for this section

- one-dimensional flow
- steady flow
- no area change
- viscous effects and wall friction do not have time to influence flow
- heat conduction and wall heat transfer do not have time to influence flow

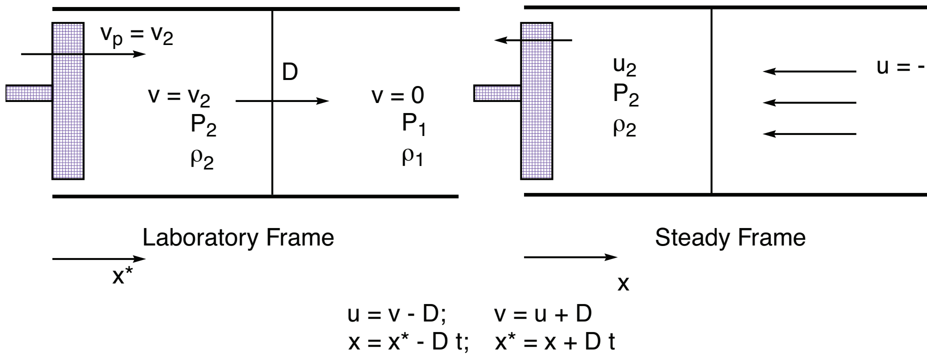

Figure 4.8: Normal shock sketch

The piston problem as sketched in Figure 4.8 will be considered.
Physical problem:

- Drive piston with known velocity v_p into fluid at rest ($v_1 = 0$) with known properties, P_1, ρ_1 in the x^* laboratory frame
- Determine disturbance speed D
- Determine disturbance properties v_2, P_2, ρ_2
- in this frame of reference **unsteady** problem

Transformed Problem:

- use Galilean transformation $x = x^* - Dt$, $u = v - D$ to transform to the frame in which the wave is at rest, therefore rending the problem **steady** in this frame
- solve as though D is known to get downstream "2" conditions: $u_2(D), P_2(D), ...$
- invert to solve for D as function of u_2, the transformed piston velocity: $D(u_2)$
- back transform to get all variables as function of v_2, the laboratory piston velocity: $D(v_2), P_2(v_2), \rho_2(v_2), ...$

4.3.1 Governing equations

Under these assumptions the conservation principles in conservative form and equation of state are in the steady frame as follows:

$$\frac{d}{dx}(\rho u) = 0 \qquad (4.267)$$

$$\frac{d}{dx}\left(\rho u^2 + P\right) = 0 \qquad (4.268)$$

$$\frac{d}{dx}\left(\rho u\left(h+\frac{u^2}{2}\right)\right) = 0 \quad (4.269)$$

$$h = h(P,\rho) \quad (4.270)$$

Upstream conditions are $\rho = \rho_1$, $P = P_1$, $u = -D$. With knowledge of the equation of state, one gets $h = h_1$. Integrating the equations from upstream to state "2" gives:

$$\rho_2 u_2 = -\rho_1 D \quad (4.271)$$

$$\rho_2 u_2^2 + P_2 = \rho_1 D^2 + P_1 \quad (4.272)$$

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{D^2}{2} \quad (4.273)$$

$$h_2 = h(P_2,\rho_2) \quad (4.274)$$

4.3.2 Rayleigh line

Work on the momentum equation:

$$P_2 = P_1 + \rho_1 D^2 - \rho_2 u_2^2 \quad (4.275)$$

$$P_2 = P_1 + \frac{\rho_1^2 D^2}{\rho_1} - \frac{\rho_2^2 u_2^2}{\rho_2} \quad (4.276)$$

Since mass gives $\rho_2^2 u_2^2 = \rho_1^2 D^2$ one gets an equation for the **Rayleigh Line**, a line in $(P, \frac{1}{\rho})$ space:

$$P_2 = P_1 + \rho_1^2 D^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \quad (4.277)$$

Note:

- Rayleigh line passes through ambient state
- Rayleigh line has *negative* slope
- magnitude of slope proportional to square of wave speed
- independent of state and energy equations

4.3.3 Hugoniot curve

Operate on the energy equation, using both mass and momentum to eliminate velocity. First eliminate u_2 via the mass equation:

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{D^2}{2} \tag{4.278}$$

$$h_2 + \frac{1}{2}\left(\frac{\rho_1 D}{\rho_2}\right)^2 = h_1 + \frac{D^2}{2} \tag{4.279}$$

$$h_2 - h_1 + \frac{D^2}{2}\left(\left(\frac{\rho_1}{\rho_2}\right)^2 - 1\right) = 0 \tag{4.280}$$

$$h_2 - h_1 + \frac{D^2}{2}\left(\frac{\rho_1^2 - \rho_2^2}{\rho_2^2}\right) = 0 \tag{4.281}$$

$$h_2 - h_1 + \frac{D^2}{2}\left(\frac{(\rho_1 - \rho_2)(\rho_1 + \rho_2)}{\rho_2^2}\right) = 0 \tag{4.282}$$

Now use the Rayleigh line to eliminate D^2:

$$D^2 = (P_2 - P_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^{-1} \tag{4.283}$$

$$D^2 = (P_2 - P_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{\rho_2 - \rho_1}{\rho_1 \rho_2}\right)^{-1} \tag{4.284}$$

$$D^2 = (P_2 - P_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right) \tag{4.285}$$

so the energy equation becomes

$$h_2 - h_1 + \frac{1}{2}(P_2 - P_1)\left(\frac{1}{\rho_1^2}\right)\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right)\left(\frac{(\rho_1 - \rho_2)(\rho_1 + \rho_2)}{\rho_2^2}\right) = 0 \tag{4.286}$$

$$h_2 - h_1 - \frac{1}{2}(P_2 - P_1)\left(\frac{1}{\rho_1}\right)\left(\frac{\rho_1 + \rho_2}{\rho_2}\right) = 0 \tag{4.287}$$

$$h_2 - h_1 - \frac{1}{2}(P_2 - P_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) = 0 \tag{4.288}$$

(4.289)

Solving finally for the enthalpy difference, one finds

$$h_2 - h_1 = (P_2 - P_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) \tag{4.290}$$

This equation is the **Hugoniot** equation.

- enthalpy change equals pressure difference times mean volume
- independent of wave speed D and velocity u_2
- independent of equation of state

4.3.4 Solution procedure for general equations of state

The shocked state can be determined by the following procedure:

- specify and equation of state $h(P, \rho)$
- substitute the equation of state into the Hugoniot to get a second relation between P_2 and ρ_2.
- use the Rayleigh line to eliminate P_2 in the Hugoniot so that the Hugoniot is a single equation in ρ_2
- solve for ρ_2 as functions of "1" and D
- back substitute to solve for P_2, u_2, h_2, T_2 as functions of "1" and D
- invert to find D as function of "1" state and u_2
- back transform to laboratory frame to get D as function of "1" state and piston velocity $v_2 = v_p$

4.3.5 Calorically perfect ideal gas solutions

Follow this procedure for the special case of a calorically perfect ideal gas.

$$h = c_p(T - T_o) + h_o \tag{4.291}$$
$$P = \rho R T \tag{4.292}$$

so

$$h = c_p \left(\frac{P}{R\rho} - \frac{P_o}{R\rho_o} \right) + h_o \tag{4.293}$$
$$h = \frac{c_p}{R} \left(\frac{P}{\rho} - \frac{P_o}{\rho_o} \right) + h_o \tag{4.294}$$
$$h = \frac{c_p}{c_p - c_v} \left(\frac{P}{\rho} - \frac{P_o}{\rho_o} \right) + h_o \tag{4.295}$$
$$h = \frac{\gamma}{\gamma - 1} \left(\frac{P}{\rho} - \frac{P_o}{\rho_o} \right) + h_o \tag{4.296}$$

Evaluate at states 1 and 2 and substitute into Hugoniot:

$$\left(\frac{\gamma}{\gamma - 1} \left(\frac{P_2}{\rho_2} - \frac{P_o}{\rho_o} \right) + h_o \right) - \left(\frac{\gamma}{\gamma - 1} \left(\frac{P_1}{\rho_1} - \frac{P_o}{\rho_o} \right) + h_o \right)$$
$$= (P_2 - P_1) \left(\frac{1}{2} \right) \left(\frac{1}{\rho_2} + \frac{1}{\rho_1} \right)$$

$$\begin{aligned}
\frac{\gamma}{\gamma-1}\left(\frac{P_2}{\rho_2}-\frac{P_1}{\rho_1}\right)-(P_2-P_1)\left(\frac{1}{2}\right)\left(\frac{1}{\rho_2}+\frac{1}{\rho_1}\right) &= 0\\
P_2\left(\frac{\gamma}{\gamma-1}\frac{1}{\rho_2}-\frac{1}{2\rho_2}-\frac{1}{2\rho_1}\right)-P_1\left(\frac{\gamma}{\gamma-1}\frac{1}{\rho_1}-\frac{1}{2\rho_2}-\frac{1}{2\rho_1}\right) &= 0\\
P_2\left(\frac{\gamma+1}{2(\gamma-1)}\frac{1}{\rho_2}-\frac{1}{2\rho_1}\right)-P_1\left(\frac{\gamma+1}{2(\gamma-1)}\frac{1}{\rho_1}-\frac{1}{2\rho_2}\right) &= 0\\
P_2\left(\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)-P_1\left(\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1}-\frac{1}{\rho_2}\right) &= 0\\
P_2 &= P_1\frac{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1}-\frac{1}{\rho_2}}{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2}-\frac{1}{\rho_1}}
\end{aligned}$$

- a *hyperbola* in $(P, \frac{1}{\rho})$ space
- $\frac{1}{\rho_2} \to \frac{\gamma-1}{\gamma+1}\frac{1}{\rho_1}$ causes $P_2 \to \infty$, note $= \gamma = 1.4, \rho_2 \to 6$ for infinite pressure
- as $\frac{1}{\rho_2} \to \infty, P_2 \to -P_1\frac{\gamma-1}{\gamma+1}$, note negative pressure, not physical here

The Rayleigh line and Hugoniot curves are sketched in Figure 4.9.

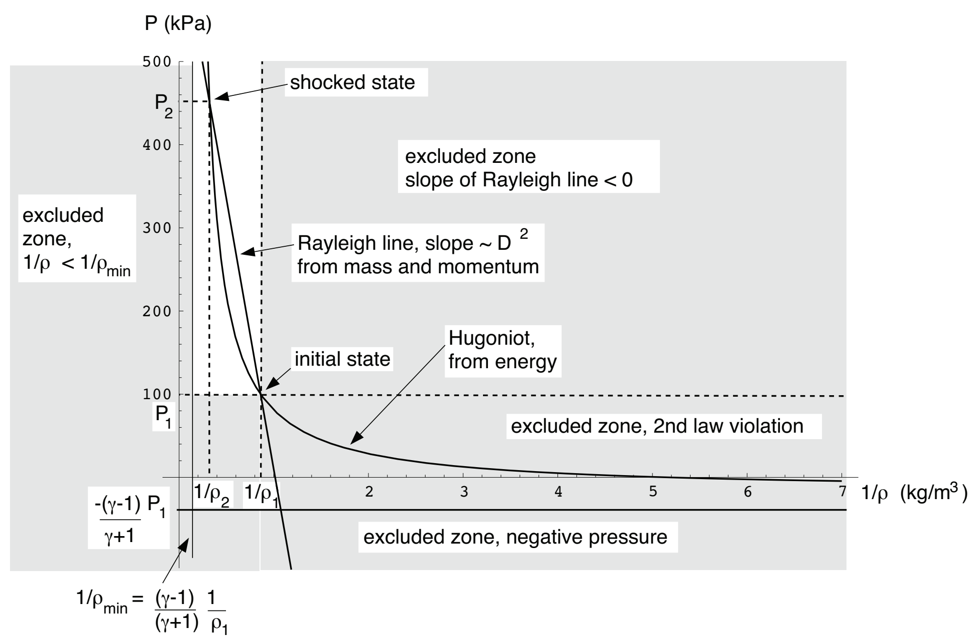

Figure 4.9: Rayleigh line and Hugoniot curve.

Note:

- intersections of the two curves are solutions to the equations
- the ambient state "1" is one solution

- the other solution “2” is known as the shock solution
- shock solution has higher pressure and higher density
- higher wave speed implies higher pressure and higher density
- a minimum wavespeed exists
  - occurs when Rayleigh line tangent to Hugoniot
  - occurs for very small pressure changes
  - corresponds to a sonic wave speed
  - disturbances are *acoustic*
- if pressure increases, can be shown entropy increases
- if pressure decreases (wave speed less than sonic), entropy decreases; this is non-physical

Substitute Rayleigh line into Hugoniot to get single equation for ρ_2

$$P_1 + \rho_1^2 D^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = P_1 \frac{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_1} - \frac{1}{\rho_2}}{\frac{\gamma+1}{\gamma-1}\frac{1}{\rho_2} - \frac{1}{\rho_1}} \tag{4.297}$$

This equation is quadratic in $\frac{1}{\rho_2}$ and factorizable. Use computer algebra to solve and get two solutions, one ambient $\frac{1}{\rho_2} = \frac{1}{\rho_1}$ and one shocked solution:

$$\frac{1}{\rho_2} = \frac{1}{\rho_1}\frac{\gamma-1}{\gamma+1}\left(1 + \frac{2\gamma}{(\gamma-1)\,D^2}\frac{P_1}{\rho_1}\right) \tag{4.298}$$

The shocked density ρ_2 is plotted against wave speed D for CPIG air in Figure 4.10.

Note

- density solution allows allows all wave speeds $0 < D < \infty$
- plot range, however, is $c_1 < D < \infty$
- Rayleigh line and Hugoniot show $D \geq c_1$
- solution for $D = D(v_p)$, *to be shown*, rigorously shows $D \geq c_1$
- *strong shock limit*: $D^2 \to \infty, \rho_2 \to \frac{\gamma+1}{\gamma-1}$
- *acoustic limit*: $D^2 \to \gamma\frac{P_1}{\rho_1}, \rho_2 \to \rho_1$
- *non-physical limit*: $D^2 \to 0, \rho_2 \to 0$

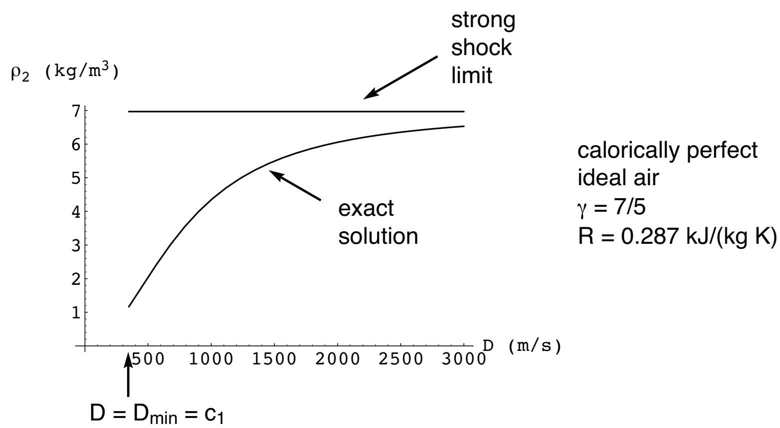

Figure 4.10: Shock density vs. shock wave speed for calorically perfect ideal air.

Back substitute into Rayleigh line and mass conservation to solve for the shocked pressure and the fluid velocity in the shocked wave frame:

$$P_2 = \frac{2}{\gamma+1}\rho_1 D^2 - \frac{\gamma-1}{\gamma+1}P_1 \tag{4.299}$$

$$u_2 = -D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)\,D^2}\frac{P_1}{\rho_1}\right) \tag{4.300}$$

The shocked pressure P_2 is plotted against wave speed D for CPIG air in Figure 4.11 including both the exact solution and the solution in the strong shock limit. Note for these parameters, the results are indistinguishable.

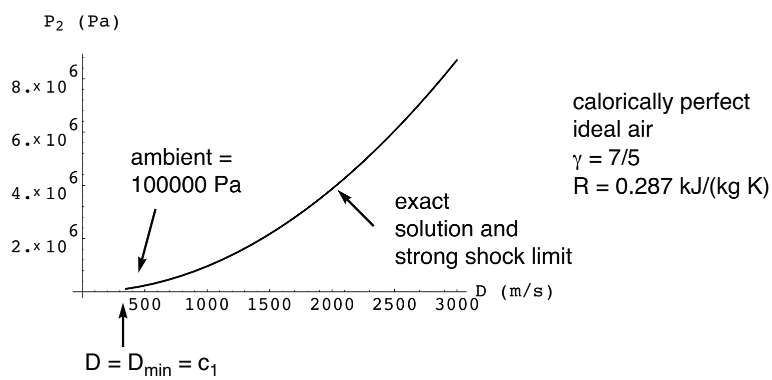

Figure 4.11: Shock pressure vs. shock wave speed for calorically perfect ideal air.

The shocked wave frame fluid particle velocity u_2 is plotted against wave speed D for CPIG air in Figure 4.12.

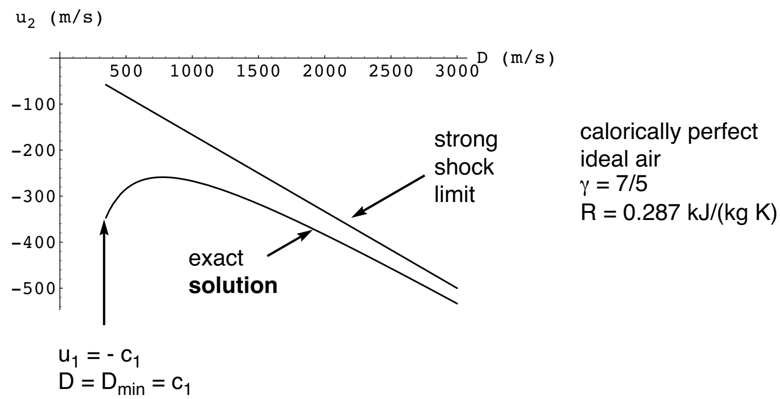

Figure 4.12: Shock wave frame fluid particle velocity vs. shock wave speed for calorically perfect ideal air.

The shocked wave frame fluid particle velocity $M_2^2 = \frac{\rho_2 u_2^2}{\gamma P_2}$ is plotted against wave speed D for CPIG air in Figure 4.13.

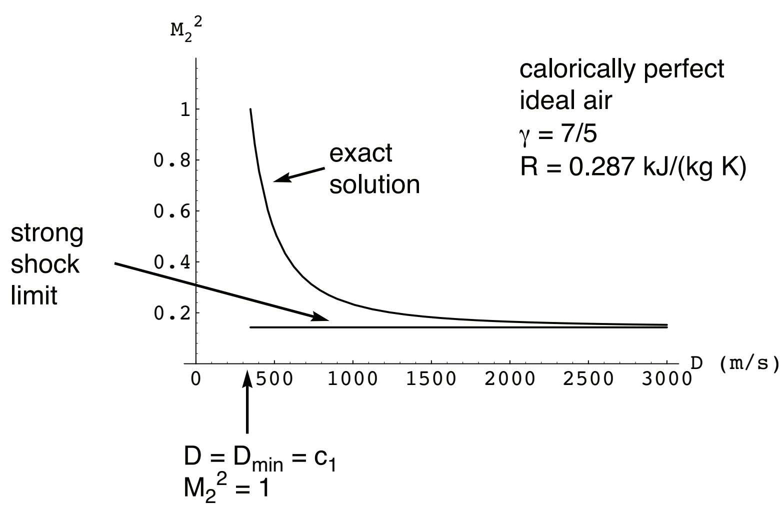

Figure 4.13: Mach number squared of shocked fluid particle vs. shock wave speed for calorically perfect ideal air.

Exercise: For the conditions shown in the plot of M_2^2 vs. D do the detailed calculations to demonstrate the plot is correct.

Note in the steady frame that

- The Mach number of the undisturbed flow is (and must be) > 1: *supersonic*
- The Mach number of the shocked flow is (and must be) < 1: *subsonic*

Transform back to the laboratory frame $u = v - D$:

$$v_2 - D = -D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)D^2}\frac{P_1}{\rho_1}\right) \quad (4.301)$$

$$v_2 = D - D\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)D^2}\frac{P_1}{\rho_1}\right) \quad (4.302)$$

Manipulate the above equation and solve the resulting quadratic equation for D and get

$$D = \frac{\gamma+1}{4}v_2 \pm \sqrt{\frac{\gamma P_1}{\rho_1} + v_2^2\left(\frac{\gamma+1}{4}\right)^2} \quad (4.303)$$

Now if $v_2 > 0$, one expects $D > 0$ so take positive root, also set velocity equal piston velocity $v_2 = v_p$

$$D = \frac{\gamma+1}{4}v_p + \sqrt{\frac{\gamma P_1}{\rho_1} + v_p^2\left(\frac{\gamma+1}{4}\right)^2} \quad (4.304)$$

Note:

- *acoustic limit*: as $v_p \to 0$, $D \to c_1$; the shock speed approaches the sound speed
- *strong shock limit*: as $v_p \to \infty, D \to \frac{\gamma+1}{2}v_p$

The shock speed D is plotted against piston velocity v_p for CPIG air in Figure 4.14. Both the exact solution and strong shock limit are shown.

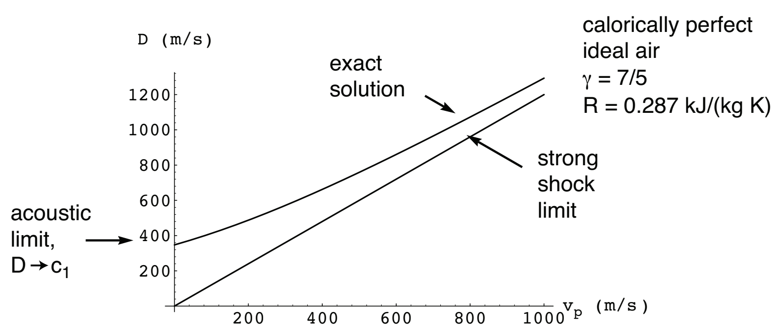

Figure 4.14: Shock speed vs. piston velocity for calorically perfect ideal air.

If the Mach number of the shock is defined as

$$M_s \equiv \frac{D}{c_1} \quad (4.305)$$

one gets

$$M_s = \frac{\gamma+1}{4}\frac{v_p}{\sqrt{\gamma RT_1}} + \sqrt{1 + \frac{v_p^2}{\gamma RT_1}\left(\frac{\gamma+1}{4}\right)^2} \tag{4.306}$$

The shock Mach number M_s is plotted against piston velocity v_p for CPIG air in Figure 4.15. Both the exact solution and strong shock limit are shown.

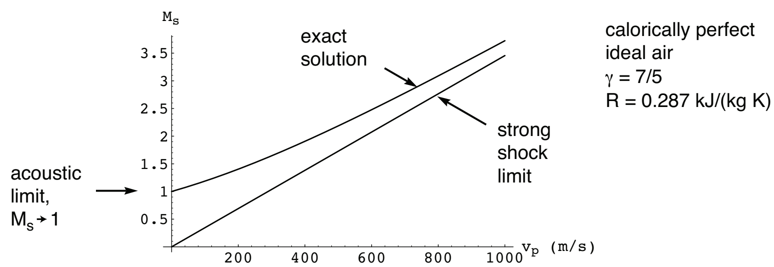

Figure 4.15: Shock Mach number vs. piston velocity for calorically perfect ideal air

Example 4.9

Normal shock problem[8]

Given: Air flowing through normal shock. Upstream $u_1 = 600\ m/s$, $T_{o1} = 500\ K$, $P_{o1} = 700\ kPa$.

Find: Downstream conditions $M_2, u_2, T_2, P_2, P_{o2}$ and $s_2 - s_1$.

Assume: calorically perfect ideal gas

Analysis:
First get all local unshocked conditions.

$$T_{o1} = T_1 + \frac{u_1^2}{2c_p} \tag{4.307}$$

$$T_1 = T_{o1} - \frac{u_1^2}{2c_p} \tag{4.308}$$

$$T_1 = 500\ K - \frac{\left(600\ \frac{m}{s}\right)^2}{2\left(1004.5\ \frac{J}{kg\ K}\right)}, \tag{4.309}$$

$$= 320.81\ K \tag{4.310}$$

$$c_1 = \sqrt{\gamma RT_1}, \tag{4.311}$$

$$= \sqrt{1.4\left(287\ \frac{J}{kg\ K}\right)(320.81\ K)}, \tag{4.312}$$

$$= 359.0\ \frac{m}{s} \tag{4.313}$$

[8] adopted from White's 9.46, p. 586

$$
\begin{aligned}
M_1 &= \frac{u_1}{c_1}, && (4.314)\\
&= \frac{600\ \frac{m}{s}}{359.0\ \frac{m}{s}}, && (4.315)\\
&= 1.671 && (4.316)\\
P_1 &= P_{o1}\left(1+\frac{1}{5}M_1^2\right)^{-3.5}, && (4.317)\\
&= (700\ kPa)\left(1+\frac{1}{5}(1.671)^2\right)^{-3.5}, && (4.318)\\
&= 148.1\ kPa && (4.319)\\
\rho_1 &= \frac{P_1}{RT_1}, && (4.320)\\
&= \frac{148.1\ kPa}{\left(0.287\ \frac{kJ}{kg\ K}\right)(320.81\ K)}, && (4.321)\\
&= 1.609\ \frac{kg}{m^3} && (4.322)\\
\frac{A_1}{A_{1*}} &= \frac{1}{M_1}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M_1^2\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} && (4.323)\\
&= \frac{1}{1.671}\left(\frac{2}{1.4+1}\left(1+\frac{1.4-1}{2}1.671^2\right)\right)^{\frac{1}{2}\frac{1.4+1}{1.4-1}}, && (4.324)\\
&= 1.311 && (4.325)
\end{aligned}
$$

Now in this case it is fortunate because the incoming velocity $D = 600\ \frac{m}{s}$ is known. Note that the shock density only depends on D^2, so one can be a little sloppy here with sign. Solve for the shocked state:

$$
\begin{aligned}
\frac{1}{\rho_2} &= \frac{1}{\rho_1}\frac{\gamma-1}{\gamma+1}\left(1+\frac{2\gamma}{(\gamma-1)D^2}\frac{P_1}{\rho_1}\right) && (4.326)\\
\frac{1}{\rho_2} &= \frac{1}{1.609\ \frac{kg}{m^3}}\frac{1.4-1}{1.4+1}\left(1+\frac{2(1.4)}{(1.4-1)\left(600\ \frac{m}{s}\right)^2}\frac{148,100\ Pa}{1.609\ \frac{kg}{m^3}}\right) && (4.327)\\
&= 0.2890\ \frac{m^3}{kg} && (4.328)\\
\rho_2 &= \frac{1}{0.2890\ \frac{m^3}{kg}}, && (4.329)\\
&= 3.461\ \frac{kg}{m^3} && (4.330)
\end{aligned}
$$

Now a variety of equations can be used to determine the remaining state variables. Mass gives u_2:

$$
\begin{aligned}
\rho_2 u_2 &= \rho_1 u_1 && (4.331)\\
u_2 &= \frac{\rho_1 u_1}{\rho_2}, && (4.332)\\
&= \frac{\left(1.609\ \frac{kg}{m^3}\right)\left(600\ \frac{m}{s}\right)}{3.461\ \frac{kg}{m^3}}, && (4.333)
\end{aligned}
$$

$$= 278.9\ \frac{m}{s}. \tag{4.334}$$

Momentum gives P_2

$$P_2 + \rho_2 u_2^2 = P_1 + \rho_1 u_1^2 \tag{4.335}$$
$$P_2 = P_1 + \rho_1 u_1^2 - \rho_2 u_2^2 \tag{4.336}$$
$$P_2 = 148,100\ Pa + \left(1.609\ \frac{kg}{m^3}\right)\left(600\ \frac{m}{s}\right)^2 - \left(3.461\ \frac{kg}{m^3}\right)\left(278.9\ \frac{m}{s}\right)^2 \tag{4.337}$$
$$P_2 = 458,125\ Pa = 458\ kPa \tag{4.338}$$

Remaining assorted variables are straightforward:

$$T_2 = \frac{P_2}{\rho_2 R} \tag{4.339}$$
$$= \frac{458,125\ Pa}{\left(3.461\ \frac{kg}{m^3}\right)\left(287\ \frac{J}{kg\ K}\right)}, \tag{4.340}$$
$$= 461.2\ K \tag{4.341}$$
$$c_2 = \sqrt{\gamma R T_2}, \tag{4.342}$$
$$= \sqrt{1.4\left(287\ \frac{J}{kg\ K}\right)(461.2\ K)}, \tag{4.343}$$
$$= 430.5\ \frac{m}{s} \tag{4.344}$$
$$M_2 = \frac{u_2}{c_2}, \tag{4.345}$$
$$= \frac{278.9\ \frac{m}{s}}{430.5\ \frac{m}{s}}, \tag{4.346}$$
$$= 0.648 \tag{4.347}$$
$$T_{o2} = T_2\left(1 + \frac{1}{5}M_2^2\right), \tag{4.348}$$
$$= 461.2\ K\left(1 + \frac{1}{5}0.648^2\right) \tag{4.349}$$
$$= 500\ K \quad \text{unchanged as required} \tag{4.350}$$
$$P_{o2} = P_2\left(1 + \frac{1}{5}M_2^2\right)^{3.5}, \tag{4.351}$$
$$= 458\ kPa\left(1 + \frac{1}{5}0.648^2\right)^{3.5} \tag{4.352}$$
$$= 607.4\ kPa \quad \text{dropped from unshocked state} \tag{4.353}$$
$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{P_2}{P_1}\right) \tag{4.354}$$
$$= 1004.5\frac{J}{kg\ K}\ln\left(\frac{461.2\ K}{320.81\ K}\right) - 287\frac{J}{kg\ K}\ln\left(\frac{458\ kPa}{148.1\ kPa}\right) \tag{4.355}$$
$$= 364.6 - 324.0, \tag{4.356}$$
$$= 40.6\ \frac{J}{kg\ K} \tag{4.357}$$
$$\frac{A_2}{A_{2*}} = \frac{1}{M_2}\left(\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M_2^2\right)\right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \tag{4.358}$$

$$= \frac{1}{0.648}\left(\frac{2}{1.4+1}\left(1+\frac{1.4-1}{2}0.648^2\right)\right)^{\frac{1}{2}\frac{1.4+1}{1.4-1}}, \quad (4.359)$$

$$= 1.966. \quad (4.360)$$

Since $A_2 = A_1 = A$,

$$\frac{A_2}{A_{2*}} = \frac{\frac{A}{A_{1*}}}{\frac{A}{A_{2*}}} = \frac{1.311}{1.966} = 0.667 \quad (4.361)$$

Note the entropy increased despite not including any entropy-generating mechanisms in this model. Why? First, the differential equations themselves required the assumption of continuous differentiable functions. Our shock violates this. When one returns to the more fundamental control volume forms, it can be shown that the entropy-generating mechanism returns. From a continuum point of view, one can also show that the neglected terms, that momentum and energy diffusion, actually give rise to a *smeared* shock. These mechanisms generate just enough entropy to satisfy the entropy jump which was just calculated. Just as with Burger's equation and the kinematic wave equation, the jumps are the same, diffusion simply gives a wave thickness.

Example 4.10

Piston Problem

Given: A piston moving at $v_p = 1,000\ \frac{m}{s}$ is driven into Helium which is at rest in the ambient state at a pressure of $P_1 = 10\ kPa$, $T_1 = 50\ K$.

Find: The shock speed and post shock state.

Assume: Helium is calorically perfect and ideal

Analysis: For Helium,

$$\gamma = 1.667 \quad (4.362)$$

$$R = 2077\ \frac{J}{kg\ K} \quad (4.363)$$

$$c_p = \frac{\gamma R}{\gamma - 1}, \quad (4.364)$$

$$= \frac{1.667\left(2,077\ \frac{J}{kg\ K}\right)}{1.667-1}, \quad (4.365)$$

$$= 5,192.5\ \frac{J}{kg\ K}. \quad (4.366)$$

Ambient density

$$\rho_1 = \frac{P_1}{RT_1}, \quad (4.367)$$

$$= \frac{10,000\ Pa}{\left(2,077\ \frac{J}{kg\ K}\right)(50\ K)}, \quad (4.368)$$

$$= 0.0963 \ \frac{kg}{m^3} \tag{4.369}$$

$$c_1 = \sqrt{\gamma R T_1}, \tag{4.370}$$

$$= \sqrt{1.667 \left(2,077 \ \frac{J}{kg \ K}\right)(50 \ K)}, \tag{4.371}$$

$$= 416.0 \ \frac{m}{s} \tag{4.372}$$

Now the wave speed D one gets from

$$D = \frac{\gamma+1}{4} v_p + \sqrt{\frac{\gamma P_1}{\rho_1} + v_p^2 \left(\frac{\gamma+1}{4}\right)^2} \tag{4.373}$$

$$= \frac{1.667+1}{4}\left(1,000 \ \frac{m}{s}\right) + \sqrt{\frac{1.667\,(10,000 \ Pa)}{0.0963 \ \frac{kg}{m^3}} + \left(1,000 \ \frac{m}{s}\right)^2 \left(\frac{1.667+1}{4}\right)^2} \tag{4.374}$$

$$= 666.7 + 785.8, \tag{4.375}$$

$$= 1,452.5 \ \frac{m}{s} \tag{4.376}$$

Strong shock limit is appropriate here as a quick check:

$$D \sim \frac{\gamma+1}{2} v_p = \frac{1.667+1}{2}\left(1,000 \ \frac{m}{s}\right) = 1,333.3 \ \frac{m}{s} \tag{4.377}$$

$$P_2 = \frac{2}{\gamma+1}\rho_1 D^2 - \frac{\gamma-1}{\gamma+1} P_1 \tag{4.378}$$

$$= \frac{2}{1.667+1}\left(0.0963 \ \frac{kg}{m^3}\right)\left(1,452.5 \ \frac{m}{s}\right)^2 - \frac{1.667-1}{1.667+1}(10,000 \ Pa) \tag{4.379}$$

$$= 152,377 - 2,500, \tag{4.380}$$

$$= 149,877 \ Pa = 150 \ kPa \tag{4.381}$$

$$\rho_2 u_2 = \rho_1 u_1 \tag{4.382}$$

$$\rho_2 (v_2 - D) = \rho_1 (v_1 - D) \tag{4.383}$$

$$\rho_2 (v_p - D) = \rho_1 (0 - D) \tag{4.384}$$

$$\rho_2 = \frac{-\rho_1 D}{v_p - D} \tag{4.385}$$

$$= \frac{-\left(0.0963 \ \frac{kg}{m^3}\right)\left(1,452.5 \ \frac{m}{s}\right)}{1,000 \ \frac{m}{s} - 1,452.5 \ \frac{m}{s}}, \tag{4.386}$$

$$= 0.309 \ \frac{kg}{m^3} \tag{4.387}$$

$$T_2 = \frac{P_2}{\rho_2 R} \tag{4.388}$$

$$= \frac{149,877 \ Pa}{\left(0.309 \ \frac{kg}{m^3}\right)\left(2,077 \ \frac{J}{kg \ K}\right)}, \tag{4.389}$$

$$= 233.5 \ K \tag{4.390}$$

4.3.6 Acoustic limit

Consider that state 2 is a small perturbation of state 1 so that

$$\rho_2 = \rho_1 + \Delta\rho \tag{4.391}$$
$$u_2 = u_1 + \Delta u_1 \tag{4.392}$$
$$P_2 = P_1 + \Delta P \tag{4.393}$$

Substituting into the normal shock equations, one gets

$$(\rho_1 + \Delta\rho)(u_1 + \Delta u) = \rho_1 u_1 \tag{4.394}$$
$$(\rho_1 + \Delta\rho)(u_1 + \Delta u)^2 + (P_1 + \Delta P) = \rho_1 {u_1}^2 + P_1 \tag{4.395}$$
$$\frac{\gamma}{\gamma - 1}\frac{P_1 + \Delta P}{\rho_1 + \Delta\rho} + \frac{1}{2}(u_1 + \Delta u)^2 = \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} + \frac{1}{2}{u_1}^2 \tag{4.396}$$

Expanding, one gets

$$\begin{aligned} \rho_1 u_1 + \tilde{u}_1(\Delta\rho) + \rho_1(\Delta u) + (\Delta\rho)(\Delta u) &= \rho_1 u_1 \\ \left(\rho_1 {u_1}^2 + 2\rho_1 u_1(\Delta u) + {u_1}^2(\Delta\rho) + \rho_1(\Delta u)^2 + 2u_1(\Delta u)(\Delta\rho) + (\Delta\rho)(\Delta u)^2\right) & \\ + (P_1 + \Delta P) &= \rho_1 {u_1}^2 + P_1 \\ \frac{\gamma}{\gamma - 1}\left(\frac{P_1}{\rho_1} + \frac{1}{\rho_1}\Delta P - \frac{P_1}{\rho_1^2}\Delta\rho + ...\right) + \frac{1}{2}\left({u_1}^2 + 2u_1(\Delta u) + (\Delta u)^2\right) & \\ &= \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} + \frac{1}{2}{u_1}^2 \end{aligned}$$

Subtracting the base state and eliminating products of small quantities yields

$$u_1(\Delta\rho) + \rho_1(\Delta u) = 0 \tag{4.397}$$
$$2\rho_1 u_1(\Delta u) + {u_1}^2(\Delta\rho) + \Delta P = 0 \tag{4.398}$$
$$\frac{\gamma}{\gamma - 1}\left(\frac{1}{\rho_1}\Delta P - \frac{P_1}{\rho_1^2}\Delta\rho\right) + u_1(\Delta u) = 0 \tag{4.399}$$

In matrix form this is

$$\begin{pmatrix} u_1 & \rho_1 & 0 \\ {u_1}^2 & 2\rho_1 u_1 & 1 \\ -\frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1^2} & u_1 & \frac{\gamma}{\gamma-1}\frac{1}{\rho_1} \end{pmatrix}\begin{pmatrix} \Delta\rho \\ \Delta u \\ \Delta P \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.400}$$

As the right hand side is zero, the determinant must be zero and there must be a linear dependency of the solution. First check the determinant:

$$u_1 \left(\frac{2\gamma}{\gamma - 1} \, u_1 - u_1 \right) - \rho_1 \left(\frac{\gamma}{\gamma - 1} \, \frac{{u_1}^2}{\rho_1} + \frac{\gamma}{\gamma - 1} \, \frac{P_1}{\rho_1^2} \right) = 0 \qquad (4.401)$$

$$\frac{{u_1}^2}{\gamma - 1} \, (2\gamma - (\gamma - 1)) - \frac{1}{\gamma - 1} \left(\gamma \, {u_1}^2 + \gamma \, \frac{P_1}{\rho_1} \right) = 0 \qquad (4.402)$$

$${u_1}^2 \, (\gamma + 1) - \left(\gamma \, {u_1}^2 + \gamma \, \frac{P_1}{\rho_1} \right) = 0 \qquad (4.403)$$

$${u_1}^2 = \gamma \frac{P_1}{\rho_1} = c_1^2 \qquad (4.404)$$

So the velocity is necessarily sonic for a small disturbance!

Take Δu to be known and solve a resulting 2×2 system:

$$\begin{pmatrix} u_1 & 0 \\ -\frac{\gamma}{\gamma-1} \, \frac{P_1}{\rho_1^2} & \frac{\gamma}{\gamma-1} \, \frac{1}{\rho_1} \end{pmatrix} \begin{pmatrix} \Delta \rho \\ \Delta P \end{pmatrix} = \begin{pmatrix} -\rho_1 \Delta u \\ -u_1 \Delta u \end{pmatrix} \qquad (4.405)$$

Solving yields

$$\Delta \rho = - \frac{\rho_1 \Delta u}{\sqrt{\gamma \, \frac{P_1}{\rho_1}}} \qquad (4.406)$$

$$\Delta P = -\rho_1 \sqrt{\gamma \, \frac{P_1}{\rho_1}} \, \Delta u \qquad (4.407)$$

4.3.7 Non-ideal gas solutions

Non-ideal effects are important

- near the critical point
- for strong shocks

Some other points:

- qualitative trends the same as for ideal gases
- analysis is much more algebraically complicated
- extraneous solutions often arise which must be discarded

Example 4.11

Shock in van der Waals gas

Given: Shock wave $D = 500 \ \frac{m}{s}$ propagating into N_2 at rest at $T_1 = 125\ K$, $P_1 = 2\ MPa$.

Find: Shocked state

Assume: van der Waals equation of state accurately models gas behavior, specific heat constant.

Analysis:

First, some data for N_2 are needed. At $P_1 = 2\ MPa$, N_2 has a boiling point of $115.5\ K$, so the material is in the gas phase but very near the vapor dome. $R = 296.8\ \frac{J}{kg\ K}$, $c_v = 744.8\ \frac{J}{kg\ K}$, $T_c = 126.2\ K$, $P_c = 3,390,000\ Pa$.

Since the material is near the vapor dome, the van der Waals equation may give a good first correction for non-ideal effects.

$$P = \frac{RT}{v-b} - \frac{a}{v^2} \tag{4.408}$$

$$P = \frac{RT}{\frac{1}{\rho} - b} - a\rho^2 \tag{4.409}$$

$$P = \frac{\rho R T}{1 - b\rho} - a\rho^2 \tag{4.410}$$

As derived earlier, the corresponding caloric equation of state is

$$e(T,v) = e_o + \int_{T_o}^{T} c_v(\hat{T}) d\hat{T} + a\left(\frac{1}{v_o} - \frac{1}{v}\right) \tag{4.411}$$

Taking c_v constant and exchanging v for ρ gives

$$e(T,\rho) = e_o + c_v\left(T - T_o\right) + a\left(\rho_o - \rho\right) \tag{4.412}$$

Eliminating T in favor of P then gives

$$e(P,\rho) = e_o + c_v\left(\frac{\left(P + a\rho^2\right)\left(1 - b\rho\right)}{\rho R} - T_o\right) + a\left(\rho_o - \rho\right) \tag{4.413}$$

and in terms of $h = e + P/\rho$:

$$h(P,\rho) = e_o + c_v\left(\frac{\left(P + a\rho^2\right)\left(1 - b\rho\right)}{\rho R} - T_o\right) + a\left(\rho_o - \rho\right) + \frac{P}{\rho} \tag{4.414}$$

and $h_2 - h_1$ allows cancellation of the “o” state so that

$$h_2 - h_1 = c_v\left(\frac{\left(P_2 + a\rho_2^2\right)\left(1 - b\rho_2\right)}{\rho_2 R} - \frac{\left(P_1 + a\rho_1^2\right)\left(1 - b\rho_1\right)}{\rho_1 R}\right) - a\left(\rho_2 - \rho_1\right) + \frac{P_2}{\rho_2} - \frac{P_1}{\rho_1} \tag{4.415}$$

The constants a and b are fixed so that an isotherm passing through the critical point, $P = P_c, T = T_c$, passes through with $\frac{\partial P}{\partial v}\big|_T = 0$ and $\frac{\partial^2 P}{\partial v^2}\Big|_T = 0$ A standard analysis[9] yields

$$\begin{aligned} a &= \frac{27}{64}\frac{R^2T_c^2}{P_c}, && (4.416)\\ &= \frac{27}{64}\frac{\left(296.8\ \frac{J}{kg\ K}\right)^2 (126.2\ K)^2}{3,390,000\ Pa}, && (4.417)\\ &= 174.6\ \frac{Pa\ m^6}{kg^2} && (4.418)\\ b &= \frac{RT_c}{8P_c}, && (4.419)\\ &= \frac{\left(296.8\ \frac{J}{kg\ K}\right)(126.2\ K)}{8\,(3,390,000\ Pa)}, && (4.420)\\ &= 0.00138\ \frac{m^3}{kg} && (4.421) \end{aligned}$$

Find the ambient density.

$$2,000,000\ Pa = \frac{\rho_1\left(296.8\ \frac{J}{kg\ K}\right)(125\ K)}{1 - \left(0.00138\ \frac{m^3}{kg}\right)\rho_1} - \left(174.6\ \frac{Pa\ m^6}{kg^2}\right)\rho_1^2 \qquad (4.422)$$

Three solutions (from computer algebra):

$$\begin{aligned} \rho_1 &= 69.0926\ \frac{kg}{m^3} \qquad \text{physical} && (4.423)\\ \rho_1 &= (327.773 + 112.702\ i)\ \frac{kg}{m^3} \qquad \text{non-physical} && (4.424)\\ \rho_1 &= (327.773 + 112.702\ i)\ \frac{kg}{m^3} \qquad \text{non-physical} && (4.425) \end{aligned}$$

Tabular data from experiments gives $\rho_1 = 71.28\ \frac{kg}{m^3}, error = (71.28-69.09)/71.28 = 3\%$, so it seems the first root is the physical root. Note that the van der Waals prediction is a significant improvement over the ideal gas law which gives $\rho_1 = \frac{P_1}{RT_1} = \frac{2,000,000}{296.8\times 125} = 53.91\ \frac{kg}{m^3}, error = (71.28-53.91)/71.28 = 21.4\%$! Even with this improvement there are much better (and more complicated!) equations of state for materials near the vapor dome.

Now use the Rayleigh line and Hugoniot equations to solve for the shocked density:

$$P_2 = P_1 + \rho_1^2 D^2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$$

$$\left(c_v\left(\frac{\left(P_2 + a\rho_2^2\right)(1 - b\rho_2)}{\rho_2 R} - \frac{\left(P_1 + a\rho_1^2\right)(1 - b\rho_1)}{\rho_1 R}\right) - a\left(\rho_2 - \rho_1\right) + \frac{P_2}{\rho_2} - \frac{P_1}{\rho_1}\right) - \left(\frac{1}{2}\right)(P_2 - P_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right) = 0$$

[9]Sonntag and Van Wylen, 1991, *Introduction to Thermodynamics: Classical and Statistical*, John Wiley: New York, p. 392.

Plugging in *all* the numbers into a computer algebra program yields the following solutions for ρ_2:

$$\rho_2 = 195.309 \frac{kg}{m^3} \quad \text{shocked solution} \tag{4.426}$$
$$\rho_2 = 69.0926 \frac{kg}{m^3} \quad \text{inert solution} \tag{4.427}$$
$$\rho_2 = (85.74 + 657.9\ i) \frac{kg}{m^3} \quad \text{non-physical solution} \tag{4.428}$$
$$\rho_2 = (85.74 - 657.9\ i) \frac{kg}{m^3} \quad \text{non-physical solution} \tag{4.429}$$

The Rayleigh line then gives the pressure:

$$P_2 = 2,000,000\ Pa + \left(69.0926 \frac{kg}{m^3}\right)^2 \left(500 \frac{m}{s}\right)^2 \left(\frac{1}{69.0926 \frac{kg}{m^3}} - \frac{1}{195.309 \frac{kg}{m^3}}\right) \tag{4.430}$$
$$P_2 = 13,162,593\ Pa = 13.2\ MPa \tag{4.431}$$

The state equation gives the temperature.

$$T_2 = \frac{\left(P_2 + a\rho_2^2\right)(1 - b\rho_2)}{\rho_2 R} \tag{4.432}$$
$$= \frac{\left(13,162,593\ Pa + \left(174.6 \frac{Pa\ m^6}{kg^2}\right)\left(195.3 \frac{kg}{m^3}\right)^2\right)\left(1 - \left(0.00138 \frac{m^3}{kg}\right)\left(195.3 \frac{kg}{m^3}\right)\right)}{\left(195.3 \frac{kg}{m^3}\right)\left(296.8 \frac{J}{kg\ K}\right)} \tag{4.433}$$
$$= 249.8\ K \tag{4.434}$$

Note the temperature is still quite low relative to standard atmospheric conditions; it is unlikely at these low temperatures that any effects due to vibrational relaxation or dissociation will be important. Our assumption of constant specific heat is probably pretty good.

The mass equation gives the shocked particle velocity:

$$\rho_2 u_2 = \rho_1 u_1 \tag{4.435}$$
$$u_2 = \frac{\rho_1 u_1}{\rho_2} \tag{4.436}$$
$$= \frac{\left(69.0926 \frac{kg}{m^3}\right)\left(500 \frac{m}{s}\right)}{195.3 \frac{kg}{m^3}} \tag{4.437}$$
$$= 176.89 \frac{m}{s} \tag{4.438}$$

An ideal gas approximation ($\gamma_{N_2} = 1.4$) would have yielded

$$\frac{1}{\rho_2} = \frac{1}{\rho_1}\frac{\gamma - 1}{\gamma + 1}\left(1 + \frac{2\gamma}{(\gamma - 1) D^2}\frac{P_1}{\rho_1}\right) \tag{4.439}$$
$$\frac{1}{\rho_2} = \left(\frac{1}{53.91 \frac{kg}{m^3}}\right)\frac{1.4 - 1}{1.4 + 1}\left(1 + \frac{2\,(1.4)}{(1.4 - 1)\left(500 \frac{m}{s}\right)^2}\frac{2,000,000\ Pa}{53.91 \frac{kg}{m^3}}\right) \tag{4.440}$$
$$\rho_2 = 158.65 \frac{kg}{m^3} \quad \text{ideal gas approximation} \tag{4.441}$$
$$relative\ error = \frac{195.3 - 158.65}{195.3} = 18.8\% \tag{4.442}$$

The Rayleigh line then gives the pressure:

$$P_2 = 2,000,000\ Pa + \left(53.91\ \frac{kg}{m^3}\right)^2 \left(500\ \frac{m}{s}\right)^2 \left(\frac{1}{53.91\ \frac{kg}{m^3}} - \frac{1}{158.65\ \frac{kg}{m^3}}\right) \quad (4.443)$$

$$P_2 = 10,897,783\ Pa = 10.90\ MPa \quad (4.444)$$

$$relative\ error = \frac{13.2 - 10.9}{13.2} = 17.4\% \quad (4.445)$$

4.4 Flow with area change and normal shocks

This section will consider flow from a reservoir with the fluid at stagnation conditions to a constant pressure environment. The pressure of the environment is commonly known as the **back pressure**: P_b.

Generic problem: Given $A(x)$, stagnation conditions and P_b, find the pressure, temperature, density at all points in the duct and the mass flow rate.

4.4.1 Converging nozzle

A converging nozzle operating at several different values of P_b is sketched in Figure 4.16.

The flow through the duct can be solved using the following procedure

- check if $P_b \geq P_*$
- if so, set $P_e = P_b$
- determine M_e from isentropic flow relations
- determine A_* from $\frac{A}{A_*}$ relation
- at any point in the flow where A is known, compute $\frac{A}{A_*}$ and then invert $\frac{A}{A_*}$ relation to find local M

Note:

- These flows are subsonic throughout and correspond to points a and b in Figure 4.16.
- If $P_b = P_*$ then the flow is sonic at the exit and just choked. This corresponds to point c in Figure 4.16.
- If $P_b < P_*$, then the flow chokes, is sonic at the exit, and continues to expand outside of the nozzle. This corresponds to points d and e in Figure 4.16.

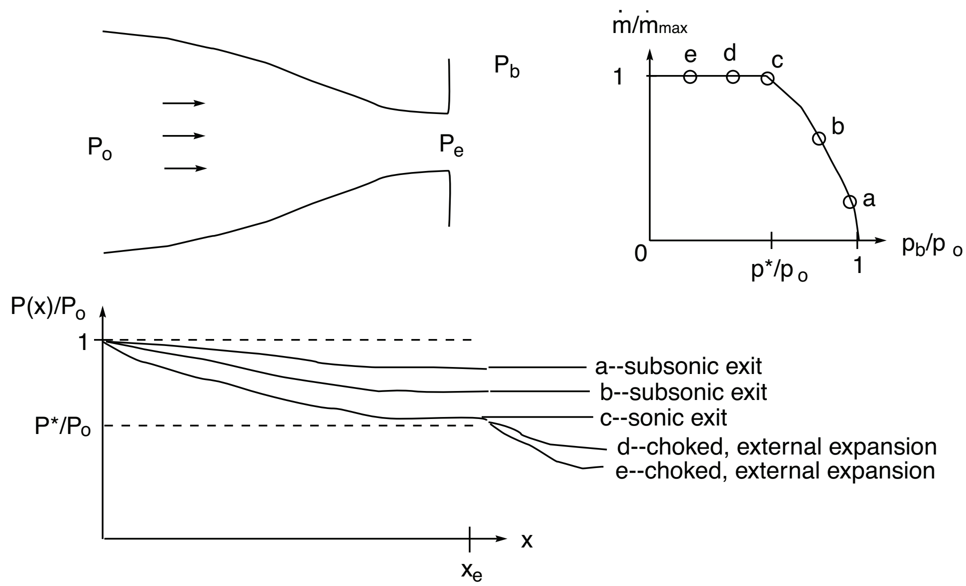

Figure 4.16: Converging nozzle sketch

4.4.2 Converging-diverging nozzle

A converging-diverging nozzle operating at several different values of P_b is sketched in Figure 4.17.

The flow through the duct can be solved using the a very similar following procedure

- set $A_t = A_*$
- with this assumption, calculate $\frac{A_e}{A_*}$
- determine M_{esub}, M_{esup}, both supersonic and subsonic, from $\frac{A}{A_*}$ relation
- determine P_{esub}, P_{esup}, from M_{esub}, M_{esup}; these are the supersonic and subsonic design pressures
- if $P_b > P_{esub}$, the flow is subsonic throughout and the throat is not sonic. Use same procedure as for converging duct: Determine M_e by setting $P_e = P_b$ and using isentropic relations
- if $P_{esub} > P_b > P_{esup}$, the procedure is complicated
  - estimate the pressure with a normal shock at the end of the duct, P_{esh}
  - If $P_b \geq P_{esh}$, there is a normal shock inside the duct
  - If $P_b < P_{esh}$, the duct flow is shockless, and there may be compression outside the duct

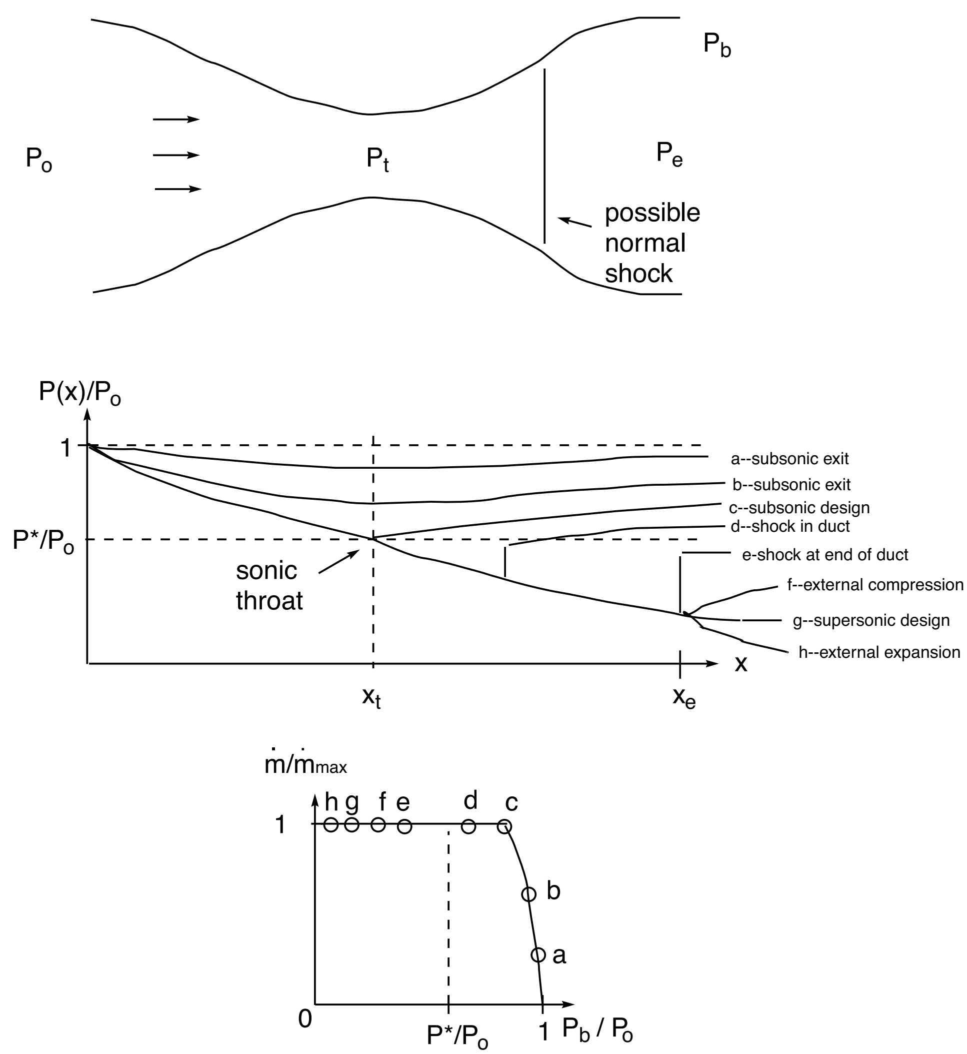

Figure 4.17: Converging-diverging nozzle sketch

- if $P_{esup} = P_b$ the flow is at supersonic design conditions and the flow is shockless
- if $P_b < P_{esup}$, the flow in the duct is isentropic and there is expansion outside the duct

Example 4.12
Nozzle Problem[10]

Given: Air at $T_o = 600\ K$ flowing through converging-diverging nozzle. $A_t = 1\ cm^2$, $A_e = 3\ cm^2$, $\dot{m} = 148.5\ \frac{kg}{hr}$. Pitot tube at exit plane gives $P_{oe} = 200\ kPa$, $P_e = 191.5\ kPa$.

[10] adopted from White's 9.69, p. 588

Find: exit velocity, location of possible normal shock in duct, Mach number just upstream of normal shock

Assume: Air is calorically perfect and ideal

Analysis:

$$\dot{m} = 148.5 \; \frac{kg}{hr} \frac{hr}{3600 \; s} = 0.04125 \; \frac{kg}{s} \tag{4.446}$$

Now if there is no shock, the stagnation pressure would be constant in the duct; one can use the choked flow formula to compare to the actual mass flow rate:

$$\dot{m}_e = \frac{P_o}{RT_o} \left(\frac{2}{\gamma + 1} \right)^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}} \sqrt{\gamma R T_o} A_* \tag{4.447}$$

$$= \frac{200,000 \; Pa}{\left(287 \; \frac{J}{kg \; K}\right) (600 \; K)} \left(\frac{2}{1.4 + 1} \right)^{\frac{1}{2}\frac{1.4+1}{1.4-1}} \sqrt{1.4 \left(287 \; \frac{J}{kg \; K}\right) (600 \; K)} \left(1 \; cm^2\right) \left(\frac{1 \; m}{100 \; cm} \right)^2 \tag{4.448}$$

$$= 200,000 \times \left(165 \times 10^{-9}\right) = 0.033 \; \frac{kg}{s} \tag{4.449}$$

Now the actual mass flow is *higher* than this, so the stagnation pressure upstream must also be higher; therefore, there must be a shock in the duct which lowers the stagnation pressure. Use this equation to determine what the upstream stagnation pressure must be.

$$0.04125 \; \frac{kg}{s} = P_{o1} \times \left(165 \times 10^{-9} \frac{kg}{s} \frac{1}{Pa} \right) \tag{4.450}$$

$$P_{o1} = 250 \; kPa \tag{4.451}$$

So

$$\frac{P_{o2}}{P_{o1}} = \frac{200 \; kPa}{250 \; kPa} = 0.800 \tag{4.452}$$

The flow conditions could be deduced from this; one can also utilize the normal shock tables for air. These are valid only for a calorically perfect ideal air. Interpolating this table yields

$$M_1 \sim 1.83 \tag{4.453}$$

$$M_2 \sim 0.61 \tag{4.454}$$

The area ratio is determined from the isentropic flow tables. Recall that A_* changes through a shock, so in this case one wants to use conditions upstream of the shock. From the tables at $M_1 = 1.83$ one finds $\frac{A_1}{A_*} = 1.4723$ so,

$$A_1 = 1.4723 \times 1 \; cm^2 = 1.4723 \; cm^2 \tag{4.455}$$

Get the exit velocity. Even if there is a shock, the stagnation temperature is constant; thus, one has from energy conservation:

$$\begin{aligned}
h_e + \frac{u_e^2}{2} &= h_o && (4.456)\\
u_e &= \sqrt{2\left(h_o - h_e\right)} && (4.457)\\
&= \sqrt{2c_p\left(T_o - T_e\right)} && (4.458)\\
&= \sqrt{2c_pT_o\left(1 - \frac{T_e}{T_o}\right)} && (4.459)\\
&= \sqrt{2c_pT_o\left(1 - \left(\frac{P_e}{P_{oe}}\right)^{\frac{\gamma-1}{\gamma}}\right)} && (4.460)\\
&= \sqrt{2\left(1004.5\ \frac{J}{kg\ K}\right)(600\ K)\left(1 - \left(\frac{191.5\ kPa}{200\ kPa}\right)^{\frac{1.4-1}{1.4}}\right)} && (4.461)\\
&= 121.9\ \frac{m}{s} && (4.462)
\end{aligned}$$

4.5 Flow with friction–Fanno flow

Wall friction is typically considered by modelling the wall shear as a constant. Wall friction is usually correlated with what is known as the *Darcy friction factor*: f, where

$$f \equiv \frac{8\tau_w}{\rho u^2} \tag{4.463}$$

Now in practice f is related to the local flow Reynolds number based on pipe diameter D: Re_D

$$Re_D \equiv \frac{\rho u D}{\mu} \tag{4.464}$$

and roughness of the duct $\frac{\epsilon}{D}$, where ϵ is the average surface roughness.

$$f = f\left(Re_D, \frac{\epsilon}{D}\right) \tag{4.465}$$

For steady laminar duct flow, the friction factor is independent of ϵ. It turns out the Poiseuille flow solution gives the friction factor, which turns out to be

$$f = \frac{64}{Re_D} \tag{4.466}$$

If the flow is steady and turbulent, the friction factor is described by the following empirical formula known as the Colebrook equation:

$$\frac{1}{f^{1/2}} = -2.0 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{Re_D f^{1/2}}\right) \tag{4.467}$$

Often one needs to iterate to find f for turbulent flows. Alternatively, one can use the **Moody chart** to estimate f. This is simply a graphical representation of the Colebrook formula. Most fluid texts will contain a Moody chart. While in principle f varies with a host of variables, in practice in a particular problem, it is often estimated as a constant.

To get a grasp on the effects of wall friction, consider a special case of generalized one-dimensional flow:

- steady
- one-dimensional
- adiabatic
- constant area duct
- Darcy friction model
- calorically perfect ideal gas

Our equations from the section on influence coefficients

$$\frac{d\rho}{dx} = \frac{1}{A}\frac{-\rho u^2 \frac{dA}{dx} + \tau_w \mathcal{L} + \frac{(q_w + \tau_w u)\mathcal{L}}{\rho u \left.\frac{\partial e}{\partial P}\right|_\rho}}{(u^2 - c^2)} \tag{4.468}$$

$$\frac{du}{dx} = \frac{1}{A}\frac{c^2 \rho u \frac{dA}{dx} - u\tau_w \mathcal{L} - \frac{(q_w + \tau_w u)\mathcal{L}}{\rho \left.\frac{\partial e}{\partial P}\right|_\rho}}{\rho\,(u^2 - c^2)} \tag{4.469}$$

$$\frac{dP}{dx} = \frac{1}{A}\frac{-c^2 \rho u^2 \frac{dA}{dx} + c^2\tau_w \mathcal{L} + \frac{(q_w + \tau_w u)\mathcal{L}u}{\rho \left.\frac{\partial e}{\partial P}\right|_\rho}}{(u^2 - c^2)} \tag{4.470}$$

reduce to

$$\frac{d\rho}{dx} = \frac{\tau_w \mathcal{L}}{A}\frac{1 + \frac{1}{\rho \left.\frac{\partial e}{\partial P}\right|_\rho}}{(u^2 - c^2)} \tag{4.471}$$

$$\frac{du}{dx} = -\frac{u\tau_w \mathcal{L}}{A}\frac{1 + \frac{1}{\rho \left.\frac{\partial e}{\partial P}\right|_\rho}}{\rho\,(u^2 - c^2)} \tag{4.472}$$

$$\frac{dP}{dx} = \frac{\tau_w \mathcal{L}}{A}\frac{c^2 + \frac{u^2}{\rho \left.\frac{\partial e}{\partial P}\right|_\rho}}{(u^2 - c^2)} \tag{4.473}$$

Now for a circular duct

$$\mathcal{L} = 2\pi r \tag{4.474}$$
$$A = \pi r^2 \tag{4.475}$$
$$\frac{\mathcal{L}}{A} = \frac{2\pi r}{\pi r^2} = \frac{2}{r} = \frac{4}{D} \tag{4.476}$$

For a calorically perfect ideal gas

$$e = \frac{1}{\gamma - 1}\frac{P}{\rho} \tag{4.477}$$
$$\left.\frac{\partial e}{\partial P}\right|_{\rho} = \frac{1}{\gamma - 1}\frac{1}{\rho} \tag{4.478}$$
$$\rho \left.\frac{\partial e}{\partial P}\right|_{\rho} = \frac{1}{\gamma - 1} \tag{4.479}$$
$$\frac{1}{\rho \left.\frac{\partial e}{\partial P}\right|_{\rho}} = \gamma - 1 \tag{4.480}$$
$$1 + \frac{1}{\rho \left.\frac{\partial e}{\partial P}\right|_{\rho}} = \gamma \tag{4.481}$$

So making these substitutions yields

$$\frac{d\rho}{dx} = \frac{4\tau_w}{D}\frac{\gamma}{(u^2 - c^2)} \tag{4.482}$$
$$\frac{du}{dx} = -\frac{4u\tau_w}{D}\frac{\gamma}{\rho\,(u^2 - c^2)} \tag{4.483}$$
$$\frac{dP}{dx} = \frac{4\tau_w}{D}\frac{c^2 + u^2\,(\gamma - 1)}{(u^2 - c^2)} \tag{4.484}$$

Substituting for τ_w gives

$$\frac{d\rho}{dx} = \frac{f\rho u^2}{2D}\frac{\gamma}{(u^2 - c^2)} \tag{4.485}$$
$$\frac{du}{dx} = -\frac{f\rho u^2 u}{2D}\frac{\gamma}{\rho\,(u^2 - c^2)} \tag{4.486}$$
$$\frac{dP}{dx} = \frac{f\rho u^2}{2D}\frac{c^2 + u^2\,(\gamma - 1)}{(u^2 - c^2)} \tag{4.487}$$

Rearranging to place in terms of M^2 gives

$$\frac{d\rho}{dx} = \frac{f\rho M^2}{2D}\frac{\gamma}{(M^2 - 1)} \tag{4.488}$$

$$\frac{du}{dx} = -\frac{fM^2u}{2D}\frac{\gamma}{(M^2-1)} \tag{4.489}$$

$$\frac{dP}{dx} = \frac{f\rho u^2}{2D}\frac{1+M^2(\gamma-1)}{(M^2-1)} \tag{4.490}$$

Now with the definition of M^2 for the calorically perfect ideal gas, one gets

$$\begin{aligned}
M^2 &= \frac{\rho u^2}{\gamma P} \\
\frac{dM^2}{dx} &= \frac{u^2}{\gamma P}\frac{d\rho}{dx} + \frac{2\rho u}{\gamma P}\frac{du}{dx} - \frac{\rho u^2}{\gamma P^2}\frac{dP}{dx} \\
&= \frac{u^2}{\gamma P}\frac{f\rho M^2}{2D}\frac{\gamma}{(M^2-1)} + \frac{2\rho u}{\gamma P}\left(-\frac{fM^2u}{2D}\frac{\gamma}{(M^2-1)}\right) - \frac{\rho u^2}{\gamma P^2}\frac{f\rho u^2}{2D}\frac{1+M^2(\gamma-1)}{(M^2-1)} \\
&= \frac{fM^4}{2D}\frac{\gamma}{(M^2-1)} - \frac{2fM^4}{2D}\frac{\gamma}{(M^2-1)} - \frac{\gamma fM^4}{2D}\frac{1+M^2(\gamma-1)}{(M^2-1)} \\
&= \frac{\gamma fM^4}{2D(M^2-1)}\left(1-2-1-M^2(\gamma-1)\right) \\
&= \frac{\gamma fM^4}{D(1-M^2)}\left(1+M^2\left(\frac{\gamma-1}{2}\right)\right)
\end{aligned}$$

So rearranging gives

$$\frac{(1-M^2)\,dM^2}{\gamma\left(M^2\right)^2\left(1+M^2\left(\frac{\gamma-1}{2}\right)\right)} = f\frac{dx}{D} \tag{4.491}$$

Integrate this expression from $x=0$ to $x=L_*$ where L_* is defined as the length at which the flow becomes sonic, so $M^2=1$ at $x=L_*$.

$$\int_{M^2}^{1}\frac{\left(1-\hat{M}^2\right)d\hat{M}^2}{\gamma\left(\hat{M}^2\right)^2\left(1+\hat{M}^2\left(\frac{\gamma-1}{2}\right)\right)} = \int_0^{L_*} f\frac{dx}{D} \tag{4.492}$$

An analytic solution for this integral is

$$\frac{1-M^2}{\gamma M^2} + \frac{1+\gamma}{2\gamma}\ln\frac{(1+\gamma)M^2}{2+M^2(\gamma-1)} = \frac{fL_*}{D} \tag{4.493}$$

Example 4.13

Flow in a duct with friction[11]

[11] from White, 9.82, p. 589

Given: Air flowing in pipe $D = 1\ in$, $L = 20\ ft$, $P_1 = 40\ psia$, $u_1 = 200\ \frac{ft}{s}$, $T_1 = 520\ R$.

Find: Exit pressure P_2, $\dot{m}$

Assume: calorically perfect ideal gas, Darcy friction factor models wall shear, constant viscosity

Analysis: First get the mass flow rate.

$$\rho_1 = \frac{P_1}{RT_1} \tag{4.494}$$

$$\rho_1 = \frac{\left(40\ \frac{lbf}{in^2}\right)\left(144\ \frac{in^2}{ft^2}\right)}{\left(53.34\ \frac{ft\ lbf}{lbm\ R}\right)(520\ R)} \tag{4.495}$$

$$\rho_1 = 0.2077\ \frac{lbm}{ft^3} \tag{4.496}$$

$$\dot{m} = \rho_1 u_1 A_1, \tag{4.497}$$

$$= \rho_1 u_1 \left(\pi \left(\frac{D}{2}\right)^2\right) \tag{4.498}$$

$$= 0.2077\ \frac{lbm}{ft^3}\left(200\ \frac{ft}{s}\right)\left(\pi\left(\frac{1\ in}{2}\frac{1\ ft}{12\ in}\right)^2\right) \tag{4.499}$$

$$= 0.2266\ \frac{lbm}{s} \tag{4.500}$$

Now compute the friction factor. First for cast iron pipes, one has surface roughness $\epsilon = 0.00085\ ft$, so

$$\frac{\epsilon}{D} = \frac{0.00085\ ft}{1\ in}\frac{12\ in}{1\ ft} = 0.0102 \tag{4.501}$$

The Reynolds number is needed, which involves the viscosity. For air at 520 R, $\mu \sim 4.08 \times 10^{-7}\ \frac{lbf\ s}{ft^2}$ so

$$Re_D = \frac{\rho_1 u_1 D}{\mu} = \frac{\left(0.2077\ \frac{lbm}{ft^3}\right)\left(200\ \frac{ft}{s}\right)\left(\frac{1}{12}\ ft\right)}{\left(4.08 \times 10^{-7}\ \frac{lbf\ s}{ft^2}\right)\left(32.17\ \frac{lbm\ ft}{lbf\ s^2}\right)} = 263,739 \tag{4.502}$$

Since $Re_D >> 2,300$, the flow is *turbulent* and one needs to use the Colebrook formula to estimate the Darcy friction factor:

$$\frac{1}{f^{1/2}} = -2.0 \log_{10}\left(\frac{\epsilon/D}{3.7} + \frac{2.51}{Re_D f^{1/2}}\right) \tag{4.503}$$

$$= -2.0 \log_{10}\left(\frac{0.0102}{3.7} + \frac{2.51}{263,739 f^{1/2}}\right) \tag{4.504}$$

Now reading the Moody chart gives $f = 0.04$. A numerical trial and error solution of the Colebrook equation gives

$$f = 0.0384 \tag{4.505}$$

Now find M_1.

$$M_1 = \frac{u_1}{\sqrt{\gamma R T_1}}, \tag{4.506}$$
$$= \frac{200\ \frac{ft}{s}}{\sqrt{1.4\left(53.34\ \frac{ft\ lbf}{lbm\ R}\right)\left(32.17\ \frac{lbm\ ft}{lbf\ s^2}\right)(520\ R)}}, \tag{4.507}$$
$$= 0.1789 \tag{4.508}$$

Now

$$\frac{fL_{1*}}{D} = \frac{1-M_1^2}{\gamma M_1^2} + \frac{1+\gamma}{2\gamma}\ln\frac{(1+\gamma)M_1^2}{2+M_1^2(\gamma-1)} \tag{4.509}$$
$$= \frac{1-0.1789^2}{1.4\,(0.1789)^2} + \frac{1+1.4}{2\,(1.4)}\ln\frac{(1+1.4)\,0.1789^2}{2+0.1789^2\,(1.4-1)} \tag{4.510}$$
$$= 18.804 \tag{4.511}$$
$$L_{1*} = \frac{18.804\left(\frac{1}{12}\ ft\right)}{0.0384}, \tag{4.512}$$
$$= 40.81\ ft \tag{4.513}$$

so at a distance 40.81 ft from station 1, the flow will go sonic. It is needed to find M_2 at a station 20 ft from station 1. So

$$L_{2*} = 40.81\ ft - 20\ ft, \tag{4.514}$$
$$= 20.81\ ft \tag{4.515}$$
$$\frac{fL_{2*}}{D} = \frac{0.0384\,(20.81\ ft)}{\frac{1}{12}\ ft}, \tag{4.516}$$
$$= 9.589 \tag{4.517}$$
$$9.589 = \frac{1-M_2^2}{1.4M_2^2} + \frac{1+1.4}{2\,(1.4)}\ln\frac{(1+1.4)\,M_2^2}{2+M_2^2\,(1.4-1)} \tag{4.518}$$

Iterative solution gives

$$M_2 = 0.237925 \tag{4.519}$$

Since energy conservation holds in this flow

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{u_1^2}{2} \tag{4.520}$$
$$T_2 + \frac{u_2^2}{2c_p} = T_1 + \frac{u_1^2}{2c_p} \tag{4.521}$$
$$T_2 + \frac{u_2^2}{2c_p} = 520\ R + \frac{\left(200\ \frac{ft}{s}\right)^2}{2\left(6,015\ \frac{ft^2}{s^2\ R}\right)} \tag{4.522}$$
$$T_2 + \frac{u_2^2}{2c_p} = 523.33\ R \tag{4.523}$$
$$T_2 + \frac{M_2^2\gamma R T_2}{2c_p} = 523.33\ R \tag{4.524}$$

$$T_2\left(1+\frac{(\gamma-1)}{2}M_2^2\right) = 523.33\ R \quad (4.525)$$

$$T_2 = \frac{523.33\ R}{1+\frac{(\gamma-1)}{2}M_2^2} \quad (4.526)$$

$$= \frac{523.33\ R}{1+\frac{(1.4-1)}{2}0.237925^2} \quad (4.527)$$

$$= 517.47\ R \quad (4.528)$$

$$u_2 = M_2\sqrt{\gamma R T_2}, \quad (4.529)$$

$$= 0.237925\sqrt{1.4\left(1,715\ \frac{ft^2}{s^2\ R}\right)(517.47\ R)} \quad (4.530)$$

$$= 265.2\ \frac{ft}{s} \quad (4.531)$$

$$\rho_2 u_2 = \rho_1 u_1 \quad (4.532)$$

$$\rho_2 = \rho_1\frac{u_1}{u_2}, \quad (4.533)$$

$$= \left(0.2077\ \frac{lbm}{ft^3}\right)\left(\frac{200\ \frac{ft}{s}}{265.2\ \frac{ft}{s}}\right), \quad (4.534)$$

$$= 0.1566\ \frac{lbm}{ft^3} \quad (4.535)$$

$$P_2 = \rho_2 R T_2, \quad (4.536)$$

$$= \left(0.1566\ \frac{lbm}{ft^3}\right)\left(53.34\ \frac{ft\ lbf}{lbm\ R}\right)(517.47\ R)\left(\frac{ft^2}{144\ in^2}\right) \quad (4.537)$$

$$= 30.02\ psia \quad (4.538)$$

$$s_2 - s_1 = c_p\ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} \quad (4.539)$$

$$= \left(6,015\ \frac{ft^2}{s^2\ R}\right)\ln\frac{517.47\ R}{520\ R} - \left(1,715\ \frac{ft^2}{s^2\ R}\right)\ln\frac{30.02\ psia}{40\ psia} \quad (4.540)$$

$$= 462.9\ \frac{ft^2}{s^2\ R} \quad (4.541)$$

4.6 Flow with heat transfer–Rayleigh flow

Flow with heat transfer is commonly known as Rayleigh flow. To isolate the effect of heat transfer, the following assumptions will be adopted:

- constant area duct
- no wall friction
- calorically perfect ideal gas

Consequences of heat addition:

- stagnation temperature changes
- heating drives both subsonic and supersonic flows **towards sonic states**
- cooling drives both subsonic and supersonic flows **away from sonic state**
- heating increases T_o, decreases P_o, both subsonic and supersonic
- cooling decreases T_o, increases P_o, both subsonic and supersonic

The governing equations are

$$\rho_2 u_2 = \rho_1 u_1 \tag{4.542}$$

$$\rho_2 u_2^2 + P_2 = \rho_1 D^2 + P_1 \tag{4.543}$$

$$\rho_2 u_2 A \left(h_2 + \frac{u_2^2}{2} \right) = \rho_1 u_1 A \left(h_1 + \frac{u_1^2}{2} \right) + q_w \mathcal{L} L \tag{4.544}$$

$$h = \frac{\gamma}{\gamma - 1} \left(\frac{P}{\rho} - \frac{P_o}{\rho_o} \right) + h_o \tag{4.545}$$

Note that these are a more general case of the equations for a normal shock. One could get equivalents of Rayleigh lines and Hugoniots. The Rayleigh line would be the same as the equations are the same; the Hugoniot would be modified because of the heat transfer term.

If one defines the heat transfer per unit mass of flow q in terms of the wall heat flux q_w:

$$q \equiv \frac{q_w \mathcal{L} L}{\rho_1 u_1 A} \tag{4.546}$$

the energy equation becomes

$$h_2 + \frac{u_2^2}{2} = h_1 + \frac{u_1^2}{2} + q \tag{4.547}$$

$$h_{o2} = h_{o1} + q \tag{4.548}$$

$$q = h_{o2} - h_{o1} \tag{4.549}$$

$$\frac{q}{c_p} = T_{o2} - T_{o1} \tag{4.550}$$

With lots of effort very similar to that used for the normal shock equations, expressions can be developed relating the "2" state to the "1" state. If one takes the final "2" state to be sonic $2 \to *$ and the initial "1" state to be unsubscripted, it is found for the calorically perfect ideal gas that

$$\frac{T_o}{T_{o*}} = \frac{(\gamma+1)\,M^2\,(2+(\gamma-1)\,M^2)}{(1+\gamma M^2)^2} \tag{4.551}$$

Example 4.14

Heat Addition Problem[12]

Given: Fuel air mixture enters combustion chamber at $u_1 = 250\ \frac{ft}{s}$, $P_1 = 20\ psia$, $T_1 = 70°F$. The mixture releases 400 $\frac{Btu}{lbm}$

Find: Exit properties u_2, P_2, T_2, heat addition to cause flow to go sonic at exit

Assume: Fuel air mixture behaves just like calorically perfect ideal air

Analysis:
Initial state

$$T_1 = 70+460, \tag{4.552}$$
$$= 530\ R \tag{4.553}$$
$$c_1 = \sqrt{\gamma R T_1}, \tag{4.554}$$
$$= \sqrt{1.4\left(1{,}716\frac{ft^2}{s^2\ R}\right)(530\ R)}, \tag{4.555}$$
$$= 1{,}128.4\ \frac{ft}{s} \tag{4.556}$$
$$M_1 = \frac{u_1}{c_1}, \tag{4.557}$$
$$= \frac{250\ \frac{ft}{s}}{1{,}128.4\ \frac{ft}{s}}, \tag{4.558}$$
$$= 0.2216 \tag{4.559}$$
$$\rho_1 = \frac{P_1}{RT_1}, \tag{4.560}$$
$$= \frac{\left(20\ \frac{lbf}{in^2}\right)\left(32.17\ \frac{lbm\ ft^2}{lbf s^2}\right)\left(144\ \frac{in^2}{ft^2}\right)}{\left(1{,}716\frac{ft^2}{s^2\ R}\right)(530\ R)}, \tag{4.561}$$
$$= 0.1019\ \frac{lbm}{ft^3} \tag{4.562}$$
$$T_{o1} = T_1\left(1+\frac{1}{5}M_1^2\right), \tag{4.563}$$
$$= (530\ R)\left(1+\frac{1}{5}0.2216^2\right), \tag{4.564}$$
$$= 535.2\ R \tag{4.565}$$
$$P_{o1} = P_1\left(1+\frac{1}{5}M_1^2\right)^{3.5}, \tag{4.566}$$

[12]adopted from White, pp. 557-558

$$= (20\ psia)\left(1+\frac{1}{5}0.2216^2\right)^{3.5}, \tag{4.567}$$
$$= 20.70\ psia \tag{4.568}$$

Now calculate T_{o*}, the stagnation temperature corresponding to sonic flow

$$\frac{T_{o1}}{T_{o*}} = \frac{(\gamma+1)M_1^2\left(2+(\gamma-1)M_1^2\right)}{\left(1+\gamma M_1^2\right)^2} \tag{4.569}$$
$$\frac{T_{o1}}{T_{o*}} = \frac{(1.4+1)\left(0.2216^2\right)\left(2+(1.4-1)\left(0.2216^2\right)\right)}{\left(1+(1.4)\left(0.2216^2\right)\right)^2}, \tag{4.570}$$
$$= 0.2084 \tag{4.571}$$
$$T_{o*} = \frac{T_{o1}}{0.2084}, \tag{4.572}$$
$$= \frac{535.2\ R}{0.2084}, \tag{4.573}$$
$$= 2568.3\ R \tag{4.574}$$

Now calculate the effect of heat addition:

$$q = \left(400\ \frac{Btu}{lbm}\right)\left(779\ \frac{ft\ lbf}{Btu}\right)\left(32.17\ \frac{lbm\ ft}{lbf\ s^2}\right), \tag{4.575}$$
$$= 10.024\times 10^6\ \frac{ft^2}{s^2} \tag{4.576}$$
$$T_{o2} = T_{o1} + \frac{q}{c_p}, \tag{4.577}$$
$$= 535.2 + \frac{10.024\times 10^6\ \frac{ft^2}{s^2}}{6,015\ \frac{ft^2}{s^2\ R}}, \tag{4.578}$$
$$= 2,201.7\ R \tag{4.579}$$
$$\frac{T_{o2}}{T_{o*}} = \frac{2,201.7\ R}{2,563.3\ R}, \tag{4.580}$$
$$= 0.8573 \tag{4.581}$$
$$\frac{T_{o2}}{T_{o*}} = \frac{(\gamma+1)M_2^2\left(2+(\gamma-1)M_2^2\right)}{\left(1+\gamma M_2^2\right)^2} \tag{4.582}$$
$$0.8573 = \frac{(1.4+1)M_2^2\left(2+(1.4-1)M_2^2\right)}{\left(1+1.4M_2^2\right)^2} \tag{4.583}$$

Computer algebra gives four solutions. For a continuous variation of M, choose the positive subsonic branch. Other branches do have physical meaning.

$$\text{relevant branch} \quad M_2 = 0.6380 \tag{4.584}$$
$$M_2 = -0.6380 \tag{4.585}$$
$$M_2 = 1.710 \tag{4.586}$$
$$M_2 = -1.710 \tag{4.587}$$

Calculate other variables at state 2:

$$T_2 = T_{o2}\left(1+\frac{1}{5}M_2^2\right)^{-1}, \tag{4.588}$$

$$
\begin{aligned}
&= (2,201.7\ R)\left(1+\frac{1}{5}\left(0.6380^2\right)\right)^{-1}, && (4.589)\\
&= 2,036\ R && (4.590)\\
c_2 &= \sqrt{\gamma R T_2}, && (4.591)\\
&= \sqrt{1.4\left(1,716\frac{ft^2}{s^2\ R}\right)(2,036\ R)}, && (4.592)\\
&= 2,211.6\ \frac{ft}{s} && (4.593)\\
u_2 &= M_2 c_2, && (4.594)\\
&= (0.6380)\left(2,211.6\ \frac{ft}{s}\right), && (4.595)\\
&= 1,411\ \frac{ft}{s} && (4.596)\\
\rho_2 u_2 &= \rho_1 u_1 && (4.597)\\
\rho_2 &= \rho_1 \frac{u_1}{u_2}, && (4.598)\\
&= \left(0.1019\ \frac{lbm}{ft^3}\right)\left(\frac{250\ \frac{ft}{s}}{1,411\ \frac{ft}{s}}\right), && (4.599)\\
&= 0.01806\ \frac{lbm}{ft^3} && (4.600)\\
P_2 &= \rho_2 R T_2 && (4.601)\\
&= \left(0.01806\ \frac{lbm}{ft^3}\right)\left(1,716\frac{ft^2}{s^2\ R}\right)\left(\frac{1}{32.17}\frac{lbf\ s^2}{lbm\ ft}\right)(2,036\ R)\frac{ft^2}{144\ in^2}, && (4.602)\\
&= 13.62\ psia && (4.603)
\end{aligned}
$$

Is momentum satisfied?

$$
\begin{aligned}
P_2 + \rho_2 u_2^2 &= P_1 + \rho_1 u_1^2\\
\left(13.62\ \frac{lbf}{in^2}\right)\left(\frac{144\ in^2}{ft^2}\right) + \left(0.01806\ \frac{lbm}{ft^3}\right)\left(1,411\ \frac{ft}{s}\right)^2\left(\frac{1}{32.17}\frac{lbf\ s^2}{lbm\ ft}\right)&\\
= \left(20\ \frac{lbf}{in^2}\right)\left(\frac{144\ in^2}{ft^2}\right) + \left(0.1019\ \frac{lbm}{ft^3}\right)\left(250\ \frac{ft}{s}\right)^2\left(\frac{1}{32.17}\frac{lbf\ s^2}{lbm\ ft}\right)&\\
3,078.97\ \frac{lbf}{ft^2} = 3077.97\ \frac{lbf}{ft^2} \qquad \text{close!}&
\end{aligned}
$$

Entropy Change

$$
\begin{aligned}
s_2 - s_1 &= c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} && (4.604)\\
&= \left(6,015\ \frac{ft^2}{s^2\ R}\right)\ln\left(\frac{2,036\ R}{530\ R}\right) - \left(1,716\ \frac{ft^2}{s^2\ R}\right)\ln\left(\frac{13.62\ psia}{20\ psia}\right) && (4.605)\\
&= 8,095.38 - (-659.28), && (4.606)\\
&= 8,754.66\ \frac{ft^2}{s^2\ R} && (4.607)\\
&= \left(8,754.66\ \frac{ft^2}{s^2\ R}\right)\left(\frac{1}{779}\frac{Btu}{ft\ lbf}\right)\left(\frac{1}{32.17}\frac{lbf\ s^2}{lbm\ ft}\right) && (4.608)
\end{aligned}
$$

$$= 0.3493 \, \frac{Btu}{lbm \; R} \tag{4.609}$$

Second Law

$$
\begin{aligned}
s_2 - s_1 &\geq \frac{q}{T} && (4.610)\\
0.3493 \, \frac{Btu}{lbm \; R} &\geq \frac{400 \, \frac{Btu}{lbm}}{2,036 \; R} && (4.611)\\
0.3493 \, \frac{Btu}{lbm \; R} &\geq 0.1965 \, \frac{Btu}{lbm \; R} \qquad \text{yes!} && (4.612)
\end{aligned}
$$

maximum heat release

$$
\begin{aligned}
q_{max} &= c_p \left(T_{o*} - T_{o1} \right) && (4.613)\\
&= \left(6,015 \, \frac{ft^2}{s^2 \; R} \right) \left(2,568.3 \; R - 535.2 \; R \right) \left(\frac{1}{779} \frac{Btu}{ft \; lbf} \right) \left(\frac{1}{32.17} \frac{lbf \; s^2}{lbm \; ft} \right) && (4.614)\\
q_{max} &= 488 \, \frac{Btu}{lbm} && (4.615)
\end{aligned}
$$

4.7 Numerical solution of the shock tube problem

A detailed development is given in lecture for the numerical solution to the Riemann or shock tube problem. The equations are first posed in the general conservative form:

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial}{\partial x} \left(\mathbf{f}(\mathbf{q}) \right) = 0. \tag{4.616}$$

Here $\mathbf{q}$ and $\mathbf{f}$ vector functions of length $N = 3$; further $\mathbf{f}$ is itself a function of $\mathbf{q}$. The equations are discretized so that

$$
\begin{aligned}
\mathbf{q}(x,t) &\rightarrow \mathbf{q}_i^n, && (4.617)\\
\mathbf{f}(\mathbf{q}(x,t)) &\rightarrow \mathbf{f}(\mathbf{q}_i^n). && (4.618)
\end{aligned}
$$

4.7.1 One-step techniques

A brief discussion of finite difference techniques is given in lecture. The most tempting technique is a first order forward difference in time, central difference in space technique which yields the finite difference relation:

$$\mathbf{q}_i^{n+1} = \mathbf{q}_i^n - \frac{\Delta t}{2 \Delta x} \left(\mathbf{f}(\mathbf{q}_{i+1}^n) - \mathbf{f}(\mathbf{q}_{i-1}^n) \right). \tag{4.619}$$

Unfortunately this method is unstable.

4.7.2 Lax-Friedrichs technique

A robustly stable first order method is found int the Lax-Friedrichs method.

$$\mathbf{q}_i^{n+1} = \frac{1}{2}\left(\mathbf{q}_{i-1}^n + \mathbf{q}_{i+1}^n\right) - \frac{\Delta t}{2\Delta x}\left(\mathbf{f}(\mathbf{q}_{i+1}^n) - \mathbf{f}(\mathbf{q}_{i-1}^n)\right). \tag{4.620}$$

4.7.3 Lax-Wendroff technique

The two-step Lax-Wendroff discretization is as follows

- at a given time step estimate $\mathbf{q}$ at the $i + 1/2$ cell interface:

$$\mathbf{q}_{i+1/2}^n = \frac{1}{2}\left(\mathbf{q}_i^n + \mathbf{q}_{i+1}^n\right), \tag{4.621}$$

- use central differencing (about $i + 1/2$) to step forward $\Delta t/2$ so that $\mathbf{q}_{i+1/2}^{n+1/2}$ can be estimated:

$$\mathbf{q}_{i+1/2}^{n+1/2} = \mathbf{q}_{i+1/2}^n - \frac{\Delta t/2}{\Delta x}\left(\mathbf{f}(\mathbf{q}_{i+1}^n) - \mathbf{f}(\mathbf{q}_i^n)\right). \tag{4.622}$$

- use central differencing (about i) to step forward Δt, evaluating $\mathbf{f}$ at the $i \pm 1/2$ and $n + 1/2$ steps:

$$\mathbf{q}_i^{n+1} = \mathbf{q}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{f}(\mathbf{q}_{i+1/2}^{n+1/2}) - \mathbf{f}(\mathbf{q}_{i-1/2}^{n+1/2})\right). \tag{4.623}$$

Chapter 5

Steady supersonic two-dimensional flow

Suggested Reading:

Liepmann and Roshko, Chapter 4: pp. 84-123
Hughes and Brighton, Chapter 8: pp. 208-230
Shapiro, Chapters 9-16: pp. 265-609
White, Chapter 9: pp. 559-581

This chapter will discuss two-dimensional flow of a compressible fluid. The following topics will be covered:

- presentation of isentropic two-dimensional flow equations
- oblique shocks
- Prandtl-Meyer rarefactions
- flow over an airfoil

Assume for this chapter:

- $\frac{\partial}{\partial t} \equiv 0$; steady flow
- $w \equiv 0, \frac{\partial}{\partial z} \equiv 0$; two-dimensional flow
- no viscous stress or heat conduction, so isentropic except through shocks
- calorically perfect ideal gas

5.1 Two-dimensional equations

With the assumptions of above the following equations govern the flow away from shock discontinuities:

5.1.1 Conservative form

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{5.1}$$

$$\frac{\partial}{\partial x}\left(\rho u^2 + P\right) + \frac{\partial}{\partial y}(\rho u v) = 0 \tag{5.2}$$

$$\frac{\partial}{\partial x}(\rho v u) + \frac{\partial}{\partial y}\left(\rho v^2 + P\right) = 0 \tag{5.3}$$

$$\frac{\partial}{\partial x}\left(\rho u\left(e + \frac{1}{2}\left(u^2 + v^2\right) + \frac{P}{\rho}\right)\right) + \frac{\partial}{\partial y}\left(\rho v\left(e + \frac{1}{2}\left(u^2 + v^2\right) + \frac{P}{\rho}\right)\right) = 0 \tag{5.4}$$

$$e = \frac{1}{\gamma - 1}\frac{P}{\rho} + e_o \tag{5.5}$$

5.1.2 Non-conservative form

$$\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{5.6}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial P}{\partial x} = 0 \tag{5.7}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial P}{\partial y} = 0 \tag{5.8}$$

$$\rho\left(u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y}\right) + P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{5.9}$$

$$e = \frac{1}{\gamma - 1}\frac{P}{\rho} + e_o \tag{5.10}$$

5.2 Mach waves

Mach waves are small acoustic disturbances in a flow field. Recall that small disturbances propagate at the ambient sound speed. Let's consider a small sphere moving at u_1 through a fluid with ambient sound speed c_o.

- $u_1 < c_o$, subsonic flow, sphere does not catch acoustic waves
- $u_1 = c_o$, sonic flow, upstream flow always unaware of sphere

- $u_1 > c_o$, supersonic flow, larger region still unaware of sphere

Consider that in time Δt, the sphere will move $u_1 \Delta t$ and the wave will propagate will be felt by a circle with radius $c_o \Delta t$, see Figure 5.1

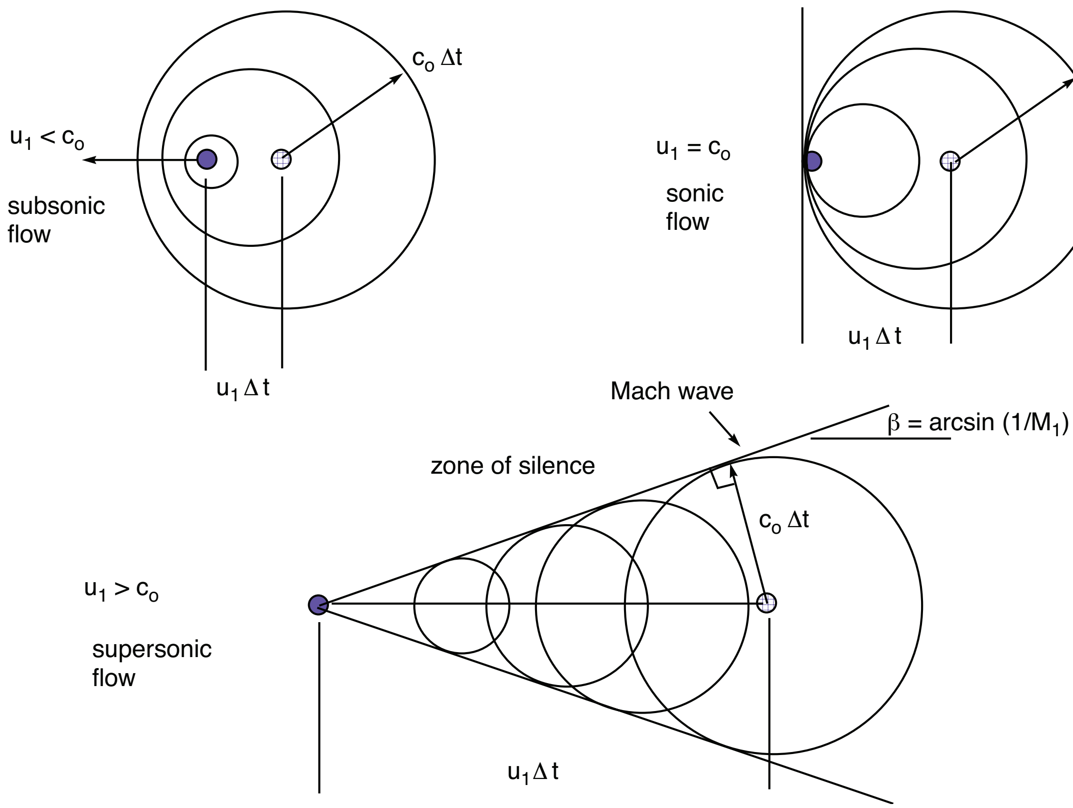

Figure 5.1: Acoustic disturbance sketch

From the geometry,

$$\sin \beta = \frac{c_o \Delta t}{u_1 \Delta t} = \frac{c_o}{u_1} = \frac{1}{M_1} \tag{5.11}$$

$$\beta = \arcsin \left(\frac{1}{M_1} \right) \tag{5.12}$$

5.3 Oblique shock waves

An oblique shock is a shock which is not normal to the incoming flow field. It can be shown that in the limiting case as the oblique shock strength goes to zero, the oblique shock wave becomes a Mach wave, as described in the previous section.

Oblique waves can be understood by considering the following problem.

Given:

- a straight wedge inclined at angle θ to the horizontal
- a freestream flow parallel to the horizontal with known velocity $\mathbf{v} = u_1\,\mathbf{i} + 0\,\mathbf{j}$
- known freestream pressure and density of P_1 and ρ_1
- steady flow of a calorically perfect ideal gas (this can be relaxed and one can still find oblique shocks)

Find:

- angle of shock inclination β
- downstream pressure and density P_2, ρ_2

Similar to the piston problem, the oblique shock problem is easiest analyzed if we instead consider

- β as known
- θ as unknown

They are best modeled in a two-dimensional coordinate system with axes parallel and perpendicular to the shock, see Figure 5.2, so that

$$x = \tilde{x}\sin\beta + \tilde{y}\cos\beta \tag{5.13}$$
$$y = -\tilde{x}\cos\beta + \tilde{y}\sin\beta \tag{5.14}$$
$$u = \tilde{u}\sin\beta + \tilde{v}\cos\beta \tag{5.15}$$
$$v = -\tilde{u}\cos\beta + \tilde{v}\sin\beta \tag{5.16}$$

Consequently, in this coordinate system, the freestream is two-dimensional.

It is easily shown that the equations of motion are invariant under a rotation of axes, so that

$$\frac{\partial}{\partial\tilde{x}}(\rho\tilde{u}) + \frac{\partial}{\partial\tilde{y}}(\rho\tilde{v}) = 0 \tag{5.17}$$
$$\frac{\partial}{\partial\tilde{x}}\left(\rho\tilde{u}^2 + P\right) + \frac{\partial}{\partial\tilde{y}}(\rho\tilde{u}\tilde{v}) = 0 \tag{5.18}$$
$$\frac{\partial}{\partial\tilde{x}}(\rho\tilde{v}\tilde{u}) + \frac{\partial}{\partial\tilde{y}}\left(\rho\tilde{v}^2 + P\right) = 0 \tag{5.19}$$
$$\frac{\partial}{\partial\tilde{x}}\left(\rho\tilde{u}\left(e + \frac{1}{2}\left(\tilde{u}^2 + \tilde{v}^2\right) + \frac{P}{\rho}\right)\right) + \frac{\partial}{\partial\tilde{y}}\left(\rho\tilde{v}\left(e + \frac{1}{2}\left(\tilde{u}^2 + \tilde{v}^2\right) + \frac{P}{\rho}\right)\right) = 0 \tag{5.20}$$
$$e = \frac{1}{\gamma - 1}\frac{P}{\rho} + e_o \tag{5.21}$$

To analyze oblique shocks, we make one *additional* assumption

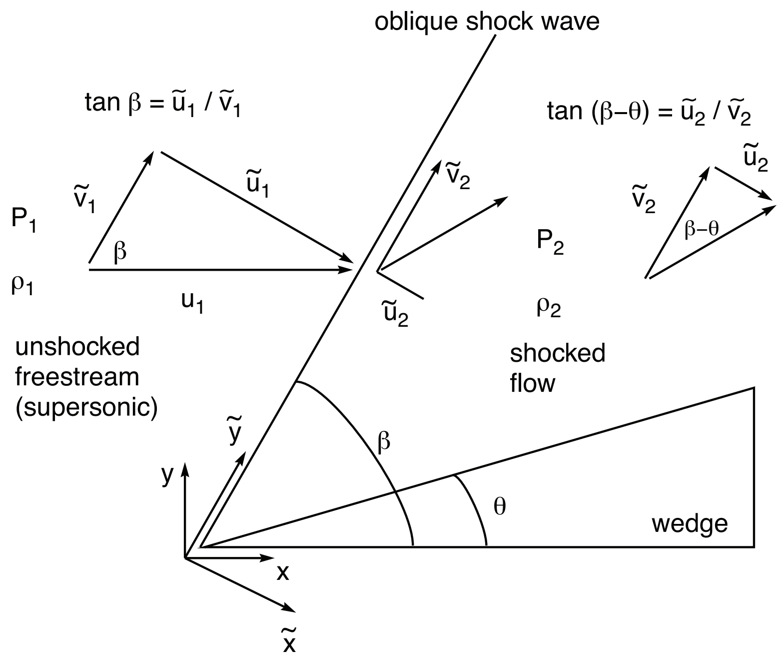

Figure 5.2: Oblique Shock Schematic

- $\frac{\partial}{\partial \tilde{y}} = 0$

Note however that, contrary to one-dimensional flow we will *not* enforce $\tilde{v} = 0$, so

- $\tilde{v} \neq 0$

Consequently, all variables are a function of $\tilde{x}$ at most and $\frac{\partial}{\partial \tilde{x}} = \frac{d}{d\tilde{x}}$. The governing equations reduce to

$$\frac{d}{d\tilde{x}}(\rho\tilde{u}) = 0 \tag{5.22}$$

$$\frac{d}{d\tilde{x}}\left(\rho\tilde{u}^2 + P\right) = 0 \tag{5.23}$$

$$\frac{d}{d\tilde{x}}(\rho\tilde{v}\tilde{u}) = 0 \tag{5.24}$$

$$\frac{d}{d\tilde{x}}\left(\rho\tilde{u}\left(e + \frac{1}{2}\left(\tilde{u}^2 + \tilde{v}^2\right) + \frac{P}{\rho}\right)\right) = 0 \tag{5.25}$$

$$e = \frac{1}{\gamma - 1}\frac{P}{\rho} + e_o \tag{5.26}$$

Integrate and apply freestream conditions

$$\rho_2 \tilde{u}_2 = \rho_1 \tilde{u}_1 \tag{5.27}$$

$$\rho_2 \tilde{u}_2^2 + P_2 = \rho_1 \tilde{u}_1^2 + P_1 \tag{5.28}$$

$$\rho_2 \tilde{v}_2 \tilde{u}_2 = \rho_1 \tilde{v}_1 \tilde{u}_1 \tag{5.29}$$

$$\rho_2 \tilde{u}_2 \left(e_2 + \frac{1}{2} \left(\tilde{u}_2^2 + \tilde{v}_2^2 \right) + \frac{P_2}{\rho_2} \right) = \rho_1 \tilde{u}_1 \left(e_1 + \frac{1}{2} \left(\tilde{u}_1^2 + \tilde{v}_1^2 \right) + \frac{P_1}{\rho_1} \right) \tag{5.30}$$

$$e = \frac{1}{\gamma - 1} \frac{P}{\rho} + e_o \tag{5.31}$$

Now using the mass equation, the $\tilde{y}$ momentum equation reduces to

$$\tilde{v}_2 = \tilde{v}_1 \tag{5.32}$$

Using this result and the mass a state equations gives

$$\rho_2 \tilde{u}_2 = \rho_1 \tilde{u}_1 \tag{5.33}$$

$$\rho_2 \tilde{u}_2^2 + P_2 = \rho_1 \tilde{u}_1^2 + P_1 \tag{5.34}$$

$$\frac{1}{\gamma - 1} \frac{P_2}{\rho_2} + \frac{1}{2} \tilde{u}_2^2 + \frac{P_2}{\rho_2} = \frac{1}{\gamma - 1} \frac{P_1}{\rho_1} + \frac{1}{2} \tilde{u}_1^2 + \frac{P_1}{\rho_1} \tag{5.35}$$

These are exactly the equations which describe a normal shock jump. All our old results apply in this coordinate system with the additional stipulation that the component of velocity *tangent* to the shock is constant.

Recall our solution for one-dimensional shocks in a calorically perfect ideal gas:

$$\frac{1}{\rho_2} = \frac{1}{\rho_1} \frac{\gamma - 1}{\gamma + 1} \left(1 + \frac{2\gamma}{(\gamma - 1) D^2} \frac{P_1}{\rho_1} \right) \tag{5.36}$$

For this problem $D = \tilde{u}_1$ so

$$\frac{1}{\rho_2} = \frac{1}{\rho_1} \frac{\gamma - 1}{\gamma + 1} \left(1 + \frac{2\gamma}{(\gamma - 1) \tilde{u}_1^2} \frac{P_1}{\rho_1} \right) \tag{5.37}$$

With the freestream Mach number *normal* to the wave defined as

$$M_{1n}^2 \equiv \frac{\tilde{u}_1^2}{\gamma \frac{P_1}{\rho_1}} \tag{5.38}$$

we get

$$\frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1}\left(1+\frac{2}{(\gamma-1)\,M_{1n}^2}\right) \tag{5.39}$$

and since from mass $\frac{\rho_1}{\rho_2} = \frac{\tilde{u}_2}{\tilde{u}_1}$

$$\frac{\tilde{u}_2}{\tilde{u}_1} = \frac{\gamma-1}{\gamma+1}\left(1+\frac{2}{(\gamma-1)\,M_{1n}^2}\right) \tag{5.40}$$

Now for our geometry

$$\tan\beta = \frac{\tilde{u}_1}{\tilde{v}_1} \tag{5.41}$$

$$\tan(\beta-\theta) = \frac{\tilde{u}_2}{\tilde{v}_2} = \frac{\tilde{u}_2}{\tilde{v}_1} \tag{5.42}$$

$$\text{so} \qquad \frac{\tilde{u}_2}{\tilde{u}_1} = \frac{\tan(\beta-\theta)}{\tan\beta} \tag{5.43}$$

thus

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{\gamma-1}{\gamma+1}\left(1+\frac{2}{(\gamma-1)\,M_{1n}^2}\right) \tag{5.44}$$

Now note that

$$M_{1n}^2 = M_1^2\sin^2\beta \tag{5.45}$$

so

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{\gamma-1}{\gamma+1}\left(1+\frac{2}{(\gamma-1)\,M_1^2\sin^2\beta}\right) \tag{5.46}$$

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{\gamma-1}{\gamma+1}\left(\frac{(\gamma-1)\,M_1^2\sin^2\beta+2}{(\gamma-1)\,M_1^2\sin^2\beta}\right) \tag{5.47}$$

$$\tan(\beta-\theta) = \tan\beta\frac{(\gamma-1)\,M_1^2\sin^2\beta+2}{(\gamma+1)\,M_1^2\sin^2\beta} \tag{5.48}$$

$$\frac{\tan\beta-\tan\theta}{1+\tan\theta\tan\beta} = \tan\beta\frac{(\gamma-1)\,M_1^2\sin^2\beta+2}{(\gamma+1)\,M_1^2\sin^2\beta} \equiv \chi \tag{5.49}$$

$$\tan\beta-\tan\theta = \chi+\chi\tan\theta\tan\beta \tag{5.50}$$

$$\tan\beta-\chi = \tan\theta\,(1+\chi\tan\beta) \tag{5.51}$$

$$\tan\theta = \frac{\tan\beta-\chi}{1+\chi\tan\beta} \tag{5.52}$$

With a little more algebra and trigonometry this reduces to

$$\tan\theta = 2\cot\beta \frac{M_1^2 \sin^2\beta - 1}{M_1^2 \left(\gamma + \cos 2\beta\right) + 2} \tag{5.53}$$

Given M_1, γ and β, this equation can be solved to find θ the wedge angle. It can be inverted to form an equation cubic in $\sin\beta$ to solve explicitly for β. Figure 5.3 gives a plot of oblique shock angle β versus wedge angle θ.

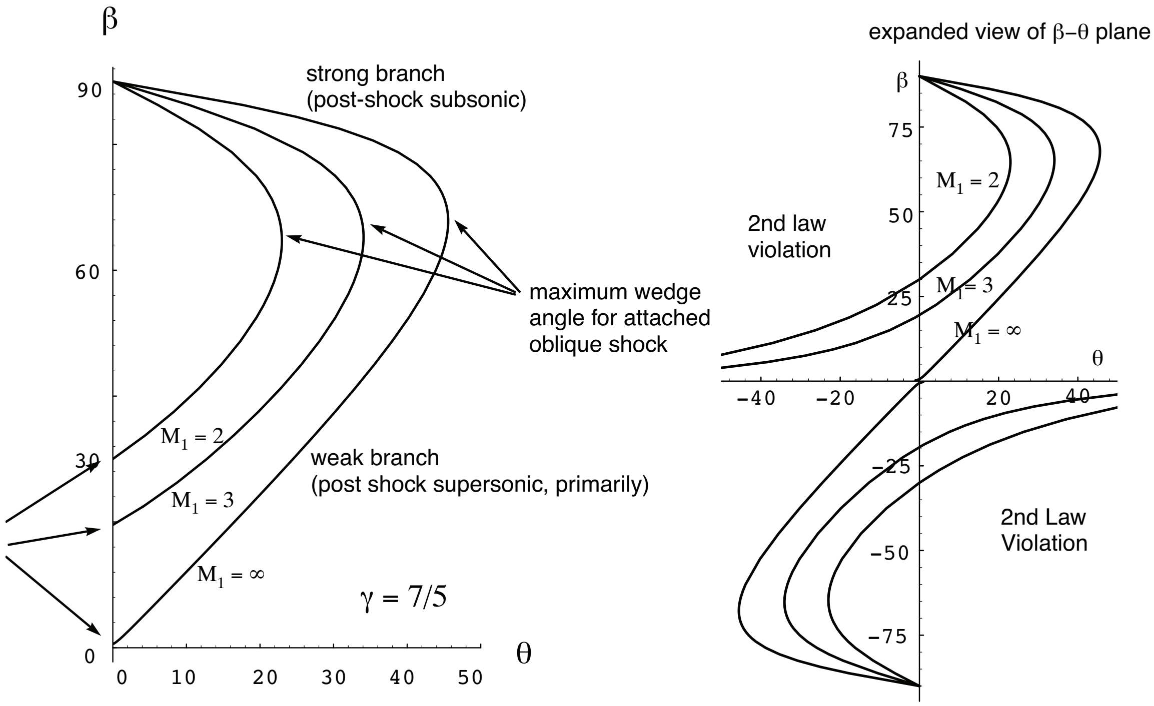

Figure 5.3: Shock angle β versus wedge angle θ

Note the following features:

- for a given $\theta < \theta_{max}$, there exist *two* β's
  - lower β is *weak* solution
    - $\lim_{\theta\to 0}\beta = \arcsin\frac{1}{M}$, a Mach wave
    - relevant branch for most external flows, matches in far-field to acoustic wave, can exist in internal flows
    - total Mach number primarily supersonic, $M_2^2 = \frac{\tilde{u}^2+\tilde{v}^2}{c_2^2} > 1$ for nearly all $0 < \theta < \theta_{max}$
    - normal Mach number subsonic, $M_{2n}^2 = \frac{\tilde{u}^2}{c_2^2} < 1$

- higher β is the *strong* solution
  - $\lim_{\theta \to 0} \beta = \frac{\pi}{2}$, a normal shock wave
  - relevant branch for some internal flows
  - total Mach number completely subsonic, $M_2^2 = \frac{\tilde{u}^2+\tilde{v}^2}{c_2^2} < 1$ for all $0 < \theta < \theta_{max}$
  - normal Mach number subsonic, $M_{2n}^2 = \frac{\tilde{u}^2}{c_2^2} < 1$

- for $\theta > \theta_{max}$, no solution exists; shock becomes detached
- Consider fixed θ, increasing freestream Mach number M_1, see Figure 5.4
  - $0 < M_1 < 1$, subsonic incoming flow, no shocks continuous pressure variation
  - $1 < M_1 < M_{1a}$, supersonic incoming flow, detached curved oblique shock
  - $M_{1a} < M_1 < \infty$, supersonic incoming flow, attached straight oblique shock
  - as $M_1 \to \infty$, $\beta \to \beta_\infty$
- Consider fixed supersonic freestream Mach number M_1, increasing θ, see Figure 5.5
  - $\theta \sim 0$, Mach wave, negligible disturbance
  - small θ, small β, small pressure, density rise
  - medium θ, medium β, moderate pressure and density rise
  - large θ, curved detached shock, large pressure and density rise

Example 5.1

Oblique Shock Example

Given: Air flowing over a wedge, $\theta = 20^\circ$, $P_1 = 100\ kPa$, $T_1 = 300\ K$, $M_1 = 3.0$

Find: Shock angle β and downstream pressure and temperature P_2, T_2.

Assume: calorically perfect ideal gas

Analysis:
First some preliminaries:

$$c_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4)\left(287\ \frac{J}{kg\ K}\right)(300\ K)} = 347.2\ \frac{m}{s} \tag{5.54}$$

$$u_1 = M_1 c_1 = (3.0)\left(347.2\ \frac{m}{s}\right) = 1,041.6\ \frac{m}{s} \tag{5.55}$$

$$\rho_1 = \frac{P_1}{RT_1} = \frac{100,000\ Pa}{\left(287\ \frac{J}{kg\ K}\right)(300\ K)} = 1.1614\ \frac{kg}{m^3} \tag{5.56}$$

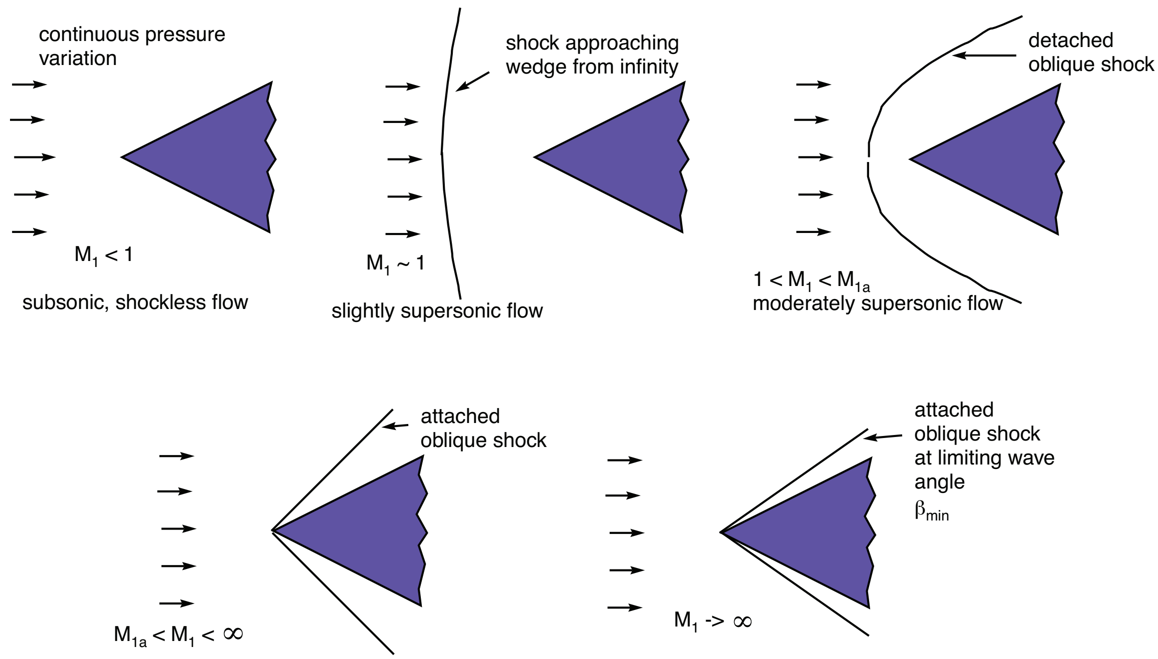

Figure 5.4: Shock wave patterns as incoming Mach number varied

Now find the wave angle:

$$\tan\theta = 2\cot\beta \frac{M_1^2 \sin^2\beta - 1}{M_1^2\left(\gamma + \cos 2\beta\right) + 2} \tag{5.57}$$

$$\tan 20^\circ = 2\cot\beta \frac{3.0^2 \sin^2\beta - 1}{3.0^2\left(1.4 + \cos 2\beta\right) + 2} \tag{5.58}$$

$$\tag{5.59}$$

Three solutions:

$$\text{weak oblique shock; common} \qquad \beta = 37.76^\circ \tag{5.60}$$

$$\text{strong oblique shock; rare} \qquad \beta = 82.15^\circ \tag{5.61}$$

$$\text{second law violating ``rarefaction'' shock} \qquad \beta = -9.91^\circ \tag{5.62}$$

1. Weak Oblique Shock

$$\tilde{u}_1 = u_1 \sin\beta = \left(1,041.6\ \frac{m}{s}\right)\sin 37.76^\circ = 637.83\ \frac{m}{s} \tag{5.63}$$

$$\tilde{v}_1 = u_1 \cos\beta = \left(1,041.6\ \frac{m}{s}\right)\cos 37.76^\circ = 823.47\ \frac{m}{s} \tag{5.64}$$

$$M_{1n} = \left(\frac{\tilde{u}_1}{c_1}\right) = \left(\frac{637.83\ \frac{m}{s}}{347.2\ \frac{m}{s}}\right) = 1.837 \tag{5.65}$$

$$\frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1}\left(1 + \frac{2}{\left(\gamma - 1\right) M_{1n}^2}\right) \tag{5.66}$$

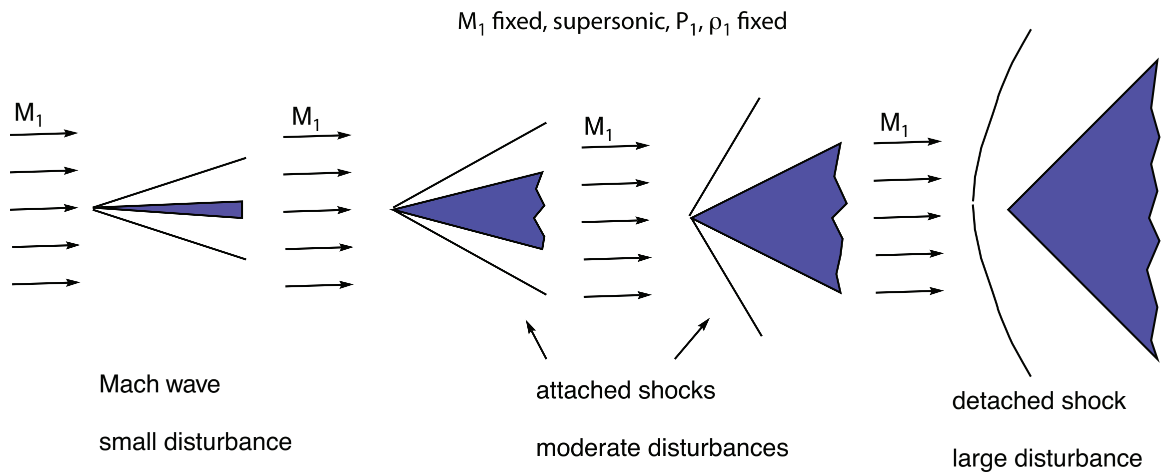

Figure 5.5: Shock wave patterns as wedge angle varied

$$\frac{1.1614\ \frac{kg}{m^3}}{\rho_2} = \frac{1.4-1}{1.4+1}\left(1+\frac{2}{(1.4-1)\,1.837^2}\right) = 0.413594 \tag{5.67}$$

$$\rho_2 = \frac{1.1614\ \frac{kg}{m^3}}{0.41359} = 2.8081\ \frac{kg}{m^3} \tag{5.68}$$

$$\rho_2\tilde{u}_2 = \rho_1\tilde{u}_1 \tag{5.69}$$

$$\tilde{u}_2 = \frac{\rho_1\tilde{u}_1}{\rho_2} = \frac{\left(1.1614\ \frac{kg}{m^3}\right)\left(637.83\ \frac{m}{s}\right)}{2.8081\ \frac{kg}{m^3}} = 263.80\ \frac{m}{s} \tag{5.70}$$

$$\tilde{v}_2 = \tilde{v}_1 = 823.47\ \frac{m}{s} \tag{5.71}$$

$$u_2 = \tilde{u}_2\sin\beta + \tilde{v}_2\cos\beta \tag{5.72}$$

$$v_2 = -\tilde{u}_2\cos\beta + \tilde{v}_2\sin\beta \tag{5.73}$$

$$u_2 = \left(263.80\ \frac{m}{s}\right)\sin 37.76^\circ + \left(823.47\ \frac{m}{s}\right)\cos 37.76^\circ = 812.56\ \frac{m}{s} \tag{5.74}$$

$$v_2 = -\left(263.80\ \frac{m}{s}\right)\cos 37.76^\circ + \left(823.47\ \frac{m}{s}\right)\sin 37.76^\circ = 295.70\ \frac{m}{s} \tag{5.75}$$

$$\text{check on wedge angle} \qquad \theta = \arctan\left(\frac{v_2}{u_2}\right) \tag{5.76}$$

$$= \arctan\left(\frac{295.70\ \frac{m}{s}}{812.56\ \frac{m}{s}}\right) = 19.997^\circ \tag{5.77}$$

$$P_2 = P_1 + \rho_1\tilde{u}_1^2 - \rho_2\tilde{u}_2^2 \tag{5.78}$$

$$P_2 = 100,000\ Pa + \left(1.1614\ \frac{kg}{m^3}\right)\left(637.83\ \frac{m}{s}\right)^2 - \left(2.8081\ \frac{kg}{m^3}\right)\left(263.80\ \frac{m}{s}\right)^2 \tag{5.79}$$

$$P_2 = 377,072\ Pa \tag{5.80}$$

$$T_2 = \frac{P_2}{\rho_2 R} = \frac{377,072\ Pa}{\left(2.8081\ \frac{kg}{m^3}\right)\left(287\ \frac{J}{kg\ K}\right)} = 467.88\ K \tag{5.81}$$

$$c_2 = \sqrt{\gamma R T_2} = \sqrt{(1.4)\left(287\ \frac{J}{kg\ K}\right)(467.88\ K)} = 433.58\ \frac{m}{s} \tag{5.82}$$

$$M_{2n} = \frac{\tilde{u}_2}{c_2} = \frac{263.8\ \frac{m}{s}}{433.58\ \frac{m}{s}} = 0.608 \tag{5.83}$$

$$M_2 = \frac{\sqrt{u_2^2 + v_2^2}}{c_2} = \frac{\sqrt{\left(812.56\ \frac{m}{s}\right)^2 + \left(295.7\ \frac{m}{s}\right)^2}}{433.58\ \frac{m}{s}} = 1.994 \tag{5.84}$$

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1} \tag{5.85}$$

$$= \left(1,004.5\ \frac{J}{kg\ K}\right)\ln\frac{467.88\ K}{300\ K} - \left(287\ \frac{J}{kg\ K}\right)\ln\frac{377,072\ Pa}{100,000\ Pa} \tag{5.86}$$

$$s_2 - s_1 = 65.50\ \frac{J}{kg\ K} \tag{5.87}$$

2. Strong Oblique Shock

$$\tilde{u}_1 = u_1 \sin\beta = \left(1,041.6\ \frac{m}{s}\right)\sin 82.15^\circ = 1,031.84\ \frac{m}{s} \tag{5.88}$$

$$\tilde{v}_1 = u_1 \cos\beta = \left(1,041.6\ \frac{m}{s}\right)\cos 82.15^\circ = 142.26\ \frac{m}{s} \tag{5.89}$$

$$M_{1n} = \left(\frac{\tilde{u}_1}{c_1}\right) = \left(\frac{1,031.84\ \frac{m}{s}}{347.2\ \frac{m}{s}}\right) = 2.972 \tag{5.90}$$

$$\frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1}\left(1 + \frac{2}{(\gamma-1)\,M_{1n}^2}\right) \tag{5.91}$$

$$\frac{1.1614\ \frac{kg}{m^3}}{\rho_2} = \frac{1.4-1}{1.4+1}\left(1 + \frac{2}{(1.4-1)\,2.972^2}\right) = 0.26102 \tag{5.92}$$

$$\rho_2 = \frac{1.1614\ \frac{kg}{m^3}}{0.26102} = 4.4495\ \frac{kg}{m^3} \tag{5.93}$$

$$\rho_2\tilde{u}_2 = \rho_1\tilde{u}_1 \tag{5.94}$$

$$\tilde{u}_2 = \frac{\rho_1\tilde{u}_1}{\rho_2} = \frac{\left(1.1614\ \frac{kg}{m^3}\right)\left(1,031.84\ \frac{m}{s}\right)}{4.4495\ \frac{kg}{m^3}} = 269.33\ \frac{m}{s} \tag{5.95}$$

$$\tilde{v}_2 = \tilde{v}_1 = 142.26\ \frac{m}{s} \tag{5.96}$$

$$u_2 = \tilde{u}_2 \sin\beta + \tilde{v}_2\cos\beta \tag{5.97}$$

$$v_2 = -\tilde{u}_2\cos\beta + \tilde{v}_2\sin\beta \tag{5.98}$$

$$u_2 = \left(269.33\ \frac{m}{s}\right)\sin 82.15^\circ + \left(142.26\ \frac{m}{s}\right)\cos 82.15^\circ = 286.24\ \frac{m}{s} \tag{5.99}$$

$$v_2 = -\left(269.33\ \frac{m}{s}\right)\cos 82.15^\circ + \left(142.26\ \frac{m}{s}\right)\sin 82.15^\circ = 104.14\ \frac{m}{s} \tag{5.100}$$

$$\text{check on wedge angle} \qquad \theta = \arctan\left(\frac{v_2}{u_2}\right) \tag{5.101}$$

$$= \arctan\left(\frac{104.14\ \frac{m}{s}}{286.24\ \frac{m}{s}}\right) = 19.99^\circ \tag{5.102}$$

$$P_2 = P_1 + \rho_1\tilde{u}_1^2 - \rho_2\tilde{u}_2^2 \tag{5.103}$$

$$P_2 = 100,000\ Pa + \left(1.1614\ \frac{kg}{m^3}\right)\left(1,031.84\ \frac{m}{s}\right)^2 - \left(4.4495\ \frac{kg}{m^3}\right)\left(269.33\ \frac{m}{s}\right)^2 \tag{5.104}$$

$$P_2 = 1,013,775\ Pa \tag{5.105}$$

$$T_2 = \frac{P_2}{\rho_2 R} = \frac{1,013,775\ Pa}{\left(4.4495\ \frac{kg}{m^3}\right)\left(287\ \frac{J}{kg\ K}\right)} = 793.86\ K \tag{5.106}$$

$$c_2 = \sqrt{\gamma R T_2} = \sqrt{(1.4)\left(287\ \frac{J}{kg\ K}\right)(793.86\ K)} = 564.78\ \frac{m}{s} \tag{5.107}$$

$$M_{2n} = \frac{\tilde{u}_2}{c_2} = \frac{269.33\ \frac{m}{s}}{564.78\ \frac{m}{s}} = 0.477 \tag{5.108}$$

$$M_2 = \frac{\sqrt{u_2^2 + v_2^2}}{c_2} = \frac{\sqrt{\left(286.24\ \frac{m}{s}\right)^2 + \left(104.14\ \frac{m}{s}\right)^2}}{564.78\ \frac{m}{s}} = 0.539 \tag{5.109}$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} \tag{5.110}$$

$$= \left(1,004.5\ \frac{J}{kg\ K}\right) \ln \frac{793.86\ K}{300\ K} - \left(287\ \frac{J}{kg\ K}\right) \ln \frac{1,013,775\ Pa}{100,000\ Pa} \tag{5.111}$$

$$s_2 - s_1 = 312.86\ \frac{J}{kg\ K} \tag{5.112}$$

3. "Rarefaction Shock"

$$\tilde{u}_1 = u_1 \sin\beta = \left(1,041.6\ \frac{m}{s}\right) \sin(-9.91^\circ) = -179.26\ \frac{m}{s} \tag{5.113}$$

$$\tilde{v}_1 = u_1 \cos\beta = \left(1,041.6\ \frac{m}{s}\right) \cos(-9.91^\circ) = 1,026.06\ \frac{m}{s} \tag{5.114}$$

$$M_{1n} = \left(\frac{\tilde{u}_1}{c_1}\right) = \left(\frac{-179.26\ \frac{m}{s}}{347.2\ \frac{m}{s}}\right) = -0.5163 \tag{5.115}$$

$$\frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1}\left(1 + \frac{2}{(\gamma - 1) M_{1n}^2}\right) \tag{5.116}$$

$$\frac{1.1614\ \frac{kg}{m^3}}{\rho_2} = \frac{1.4 - 1}{1.4 + 1}\left(1 + \frac{2}{(1.4 - 1)(-0.5163)^2}\right) = 3.2928 \tag{5.117}$$

$$\rho_2 = \frac{1.1614\ \frac{kg}{m^3}}{3.2928} = 0.3527\ \frac{kg}{m^3} \tag{5.118}$$

$$\rho_2 \tilde{u}_2 = \rho_1 \tilde{u}_1 \tag{5.119}$$

$$\tilde{u}_2 = \frac{\rho_1 \tilde{u}_1}{\rho_2} = \frac{\left(1.1614\ \frac{kg}{m^3}\right)\left(-179.26\ \frac{m}{s}\right)}{0.3527\ \frac{kg}{m^3}} = -590.27\ \frac{m}{s} \tag{5.120}$$

$$\tilde{v}_2 = \tilde{v}_1 = 1,026.06\ \frac{m}{s} \tag{5.121}$$

$$u_2 = \tilde{u}_2 \sin\beta + \tilde{v}_2 \cos\beta \tag{5.122}$$

$$v_2 = -\tilde{u}_2 \cos\beta + \tilde{v}_2 \sin\beta \tag{5.123}$$

$$u_2 = \left(-590.27\ \frac{m}{s}\right) \sin(-9.91^\circ) + \left(1,026.06\ \frac{m}{s}\right) \cos(-9.91^\circ) = 1,112.34\ \frac{m}{s} \tag{5.124}$$

$$v_2 = -\left(-590.27\ \frac{m}{s}\right) \cos(-9.91^\circ) + \left(1,026.06\ \frac{m}{s}\right) \sin(-9.91^\circ) = 404.88\ \frac{m}{s} \tag{5.125}$$

$$\text{check on wedge angle} \qquad \theta = \arctan\left(\frac{v_2}{u_2}\right) \tag{5.126}$$

$$= \arctan\left(\frac{404.88\ \frac{m}{s}}{1,112.34\ \frac{m}{s}}\right) = 20.00^\circ \tag{5.127}$$

$$P_2 = P_1 + \rho_1 \tilde{u}_1^2 - \rho_2 \tilde{u}_2^2 \tag{5.128}$$

$$P_2 = 100,000\ Pa + \left(1.1614\ \frac{kg}{m^3}\right)\left(-179.26\ \frac{m}{s}\right)^2 - \left(0.3527\ \frac{kg}{m^3}\right)\left(-590.27\ \frac{m}{s}\right)^2 \tag{5.129}$$

$$P_2 = 14,433\ Pa \tag{5.130}$$

$$T_2 = \frac{P_2}{\rho_2 R} = \frac{14,433\ Pa}{\left(0.3527\ \frac{kg}{m^3}\right)\left(287\ \frac{J}{kg\ K}\right)} = 142.59\ K \tag{5.131}$$

$$c_2 = \sqrt{\gamma R T_2} = \sqrt{(1.4)\left(287\ \frac{J}{kg\ K}\right)(142.59\ K)} = 239.36\ \frac{m}{s} \tag{5.132}$$

$$M_{2n} = \frac{\tilde{u}_2}{c_2} = \frac{-590.27\ \frac{m}{s}}{239.36\ \frac{m}{s}} = -2.47 \tag{5.133}$$

$$M_2 = \frac{\sqrt{u_2^2 + v_2^2}}{c_2} = \frac{\sqrt{\left(1,112.34\ \frac{m}{s}\right)^2 + \left(404.88\ \frac{m}{s}\right)^2}}{239.36\ \frac{m}{s}} = 4.95 \tag{5.134}$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} \tag{5.135}$$

$$= \left(1,004.5\ \frac{J}{kg\ K}\right) \ln \frac{142.59\ K}{300\ K} - \left(287\ \frac{J}{kg\ K}\right) \ln \frac{14,433\ Pa}{100,000\ Pa} \tag{5.136}$$

$$s_2 - s_1 = -191.5\ \frac{J}{kg\ K} \tag{5.137}$$

5.4 Small disturbance theory

By taking a Taylor series expansion of the relationship between β and θ about $\theta = 0$, for fixed M_1 and γ it can be shown that

$$\tan \beta = \frac{1}{\sqrt{M_1^2 - 1}} + \frac{\gamma + 1}{4} \frac{M_1^4}{\left(M_1^2 - 1\right)^2}\,\theta \qquad \theta << 1 \tag{5.138}$$

Note that when $\theta = 0$ that

$$\tan \beta = \frac{1}{\sqrt{M_1^2 - 1}} \tag{5.139}$$

$$\tan^2 \beta = \frac{1}{M_1^2 - 1} \tag{5.140}$$

$$\frac{\sin^2 \beta}{\cos^2 \beta} = \frac{\sin^2 \beta}{1 - \sin^2 \beta} = \frac{1}{M_1^2 - 1} \tag{5.141}$$

$$\frac{\sin^2 \beta}{1 - \sin^2 \beta} = \frac{\frac{1}{M_1^2}}{1 - \frac{1}{M_1^2}} \tag{5.142}$$

$$\sin\beta = \frac{1}{M_1} \tag{5.143}$$

$$\beta = \arcsin\left(\frac{1}{M_1}\right) \tag{5.144}$$

After a good deal of algebra and trigonometry, it can also be shown that the pressure change, change in *velocity magnitude*, w, and change in entropy for flow over a thin wedge is

$$\frac{P_2 - P_1}{P_1} = \frac{\gamma M_1^2}{\sqrt{M_1^2 - 1}}\,\theta \tag{5.145}$$

$$\frac{w_2 - w_1}{w_1} = -\frac{\theta}{\sqrt{M_1^2 - 1}} \tag{5.146}$$

$$\frac{s_2 - s_1}{s_1} \sim \theta^3 \tag{5.147}$$

In terms of changes,

$$\frac{\Delta P}{P_1} = \frac{\gamma M_1^2}{\sqrt{M_1^2 - 1}}\,\Delta\theta \tag{5.148}$$

$$\frac{\Delta w}{w_1} = -\frac{\Delta\theta}{\sqrt{M_1^2 - 1}} \tag{5.149}$$

$$\frac{\Delta s}{s_1} \sim \Delta\theta^3 \tag{5.150}$$

Note that a small positive $\Delta\theta$ gives rise to

- an increase in pressure
- a decrease in velocity magnitude
- a very small change in entropy

Figure 5.6 shows the pattern of waves that one obtains when subjecting a flow to a series of small turns and the pattern that evolves as the turning radius is shrunk.

- compression waves converge
- expansion waves diverge
- convergence of compression waves leads to region of rapid entropy rise–shock formation
- divergence of pressure waves leads to no shock formation in expansion

This has an analog in one-dimensional unsteady flow. Consider a piston with an initial velocity of zero accelerating into a tube

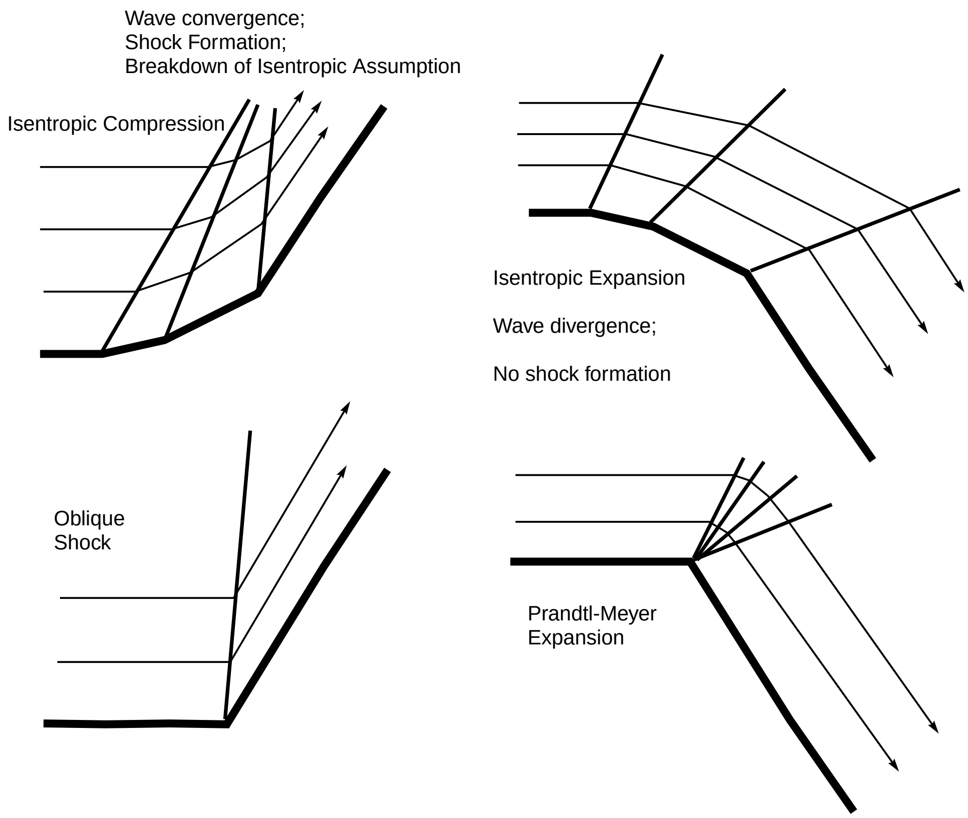

Figure 5.6: Wave pattern and streamlines for flows undergoing a series of small turns and for sudden turns

- lead compression wave travels at sound speed
- lead wave increases temperature (and sound speed) of disturbed flow
- each successive acoustic wave travels faster than lead wave
- eventually acoustic waves catch and form a shock

Consider a piston with zero initial velocity which decelerates

- lead expansion wave travels at sound speed
- lead wave decreases temperature (and sound speed) of disturbed flow
- each successive acoustic wave travels slow than lead wave
- no shock formation

A schematic for these one-dimensional unsteady flows is shown in Figure 5.7

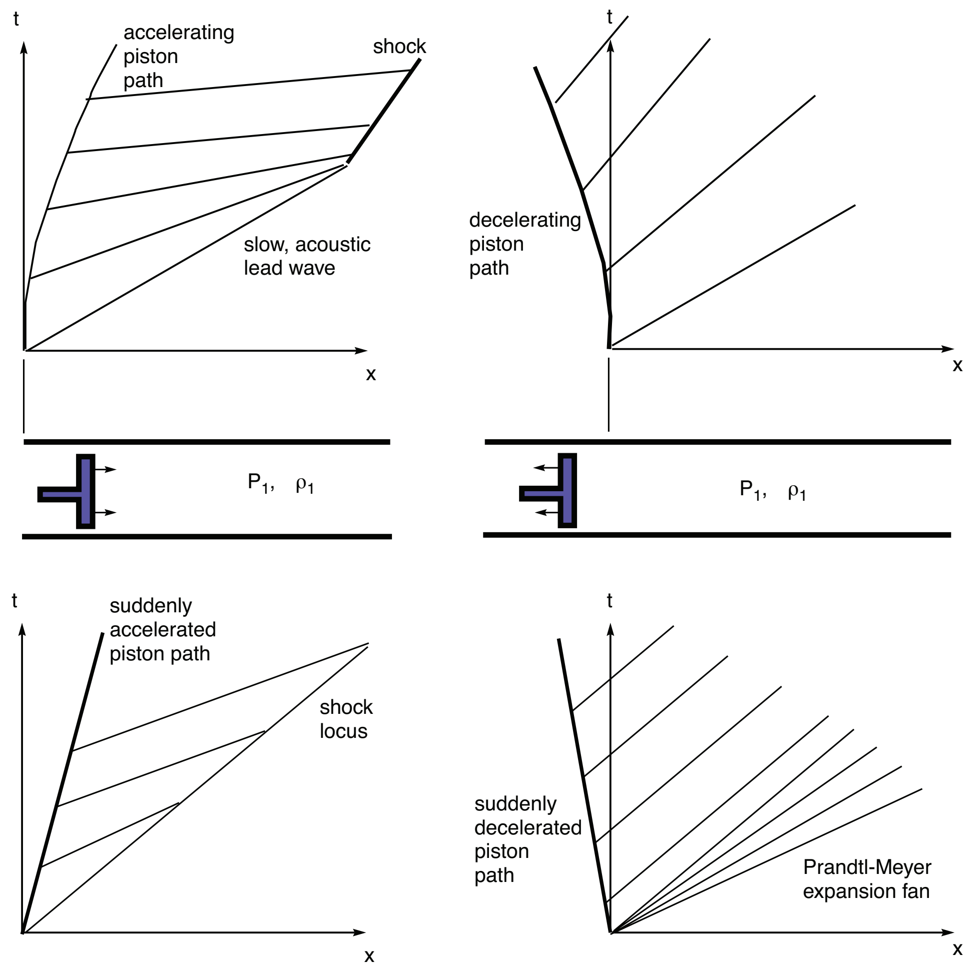

Figure 5.7: Schematic of compression and expansion waves for one-dimensional unsteady piston-driven flow

5.5 Centered Prandtl-Meyer rarefaction

If we let $\Delta\theta \to 0$, the entropy changes become negligibly small relative to pressure and velocity changes, and the flow is isentropic. The relations can be replaced by differential relations:

$$\frac{dP}{P} = \frac{\gamma M^2}{\sqrt{M^2-1}}\, d\theta \tag{5.151}$$

$$\frac{dw}{w} = -\frac{d\theta}{\sqrt{M^2-1}} \tag{5.152}$$

$$\frac{ds}{s} \sim 0 \tag{5.153}$$

Recall now that for adiabatic flow

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{5.154}$$

$$\frac{\gamma R T_o}{\gamma R T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{5.155}$$

$$\frac{c_o^2}{c^2} = 1 + \frac{\gamma - 1}{2} M^2 \tag{5.156}$$

$$c = c_o \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{2}} \tag{5.157}$$

$$dc = -\frac{c_o}{2} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{3}{2}} (\gamma - 1) M dM \tag{5.158}$$

$$\frac{dc}{c} = -\frac{\frac{\gamma-1}{2} M dM}{1 + \frac{\gamma-1}{2} M^2} \tag{5.159}$$

Also

$$w = cM \tag{5.160}$$

$$dw = c dM + M dc \tag{5.161}$$

$$\frac{dw}{w} = \frac{dM}{M} + \frac{dc}{c} \tag{5.162}$$

$$\frac{dw}{w} = \frac{dM}{M} - \frac{\frac{\gamma-1}{2} M dM}{1 + \frac{\gamma-1}{2} M^2} \tag{5.163}$$

$$\frac{dw}{w} = \frac{1}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M} \tag{5.164}$$

$$-\frac{d\theta}{\sqrt{M^2 - 1}} = \frac{1}{1 + \frac{\gamma-1}{2} M^2} \frac{dM}{M} \tag{5.165}$$

$$-d\theta = \frac{\sqrt{M^2 - 1}}{M} \frac{dM}{1 + \frac{\gamma-1}{2} M^2} \tag{5.166}$$

Now positive θ corresponds to compression and negative θ corresponds to expansion. Let's define ν so positive ν gives and expansion.

$$\nu \equiv -\theta + \theta_o \tag{5.167}$$

$$d\nu = -d\theta \tag{5.168}$$

Now integrate the expression

$$d\nu = \frac{\sqrt{M^2 - 1}}{M} \frac{dM}{1 + \frac{\gamma-1}{2} M^2} \tag{5.169}$$

Let $\nu = 0$ correspond to $M = 1$. This effectively selects θ_o

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1} \tag{5.170}$$

The function $\nu(M)$ is called the **Prandtl-Meyer function**. It is plotted in Figure 5.8. Many texts tabulate the Prandtl-Meyer function. For a known turning angle, one can find the Mach number. As the flow is entirely isentropic, all other flow variables can be obtained through the isentropic relations. Note:

- As $M \to \infty$, $\nu \to \frac{\pi}{2}\left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right)$, corresponds to vacuum conditions
- given ν, one can calculate M
- isentropic relations give P, ρ, T, etc.
- Prandtl-Meyer function tabulated in many texts

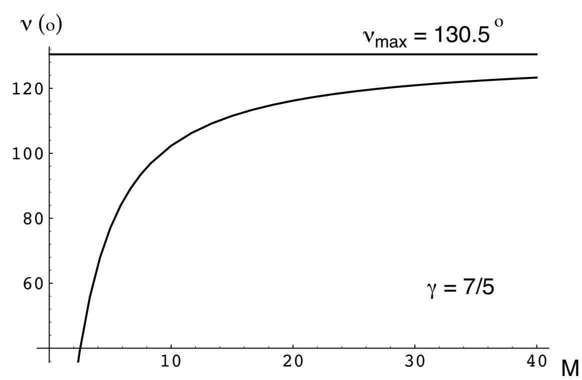

Figure 5.8: Prandtl-Meyer function

Example 5.2

Centered Expansion

Given: Calorically perfect, ideal air with $P_1 = 100\ kPa$, $T_1 = 300\ K$, $u_1 = 500\ \frac{m}{s}$, turned through a 30° expansion corner.

Find: Fluid properties after the expansion

Analysis:

$$\rho_1 = \frac{P_1}{RT_1} = \frac{100\ kPa}{\left(0.287\ \frac{J}{kg\ K}\right)(300\ K)} = 1.1614\ \frac{kg}{m^3} \tag{5.171}$$

$$c_1 = \sqrt{\gamma R T_1} = \sqrt{1.4\left(287\ \frac{J}{kg\ K}\right)(300\ K)} = 347.2\ \frac{m}{s} \tag{5.172}$$

$$M_1 = \frac{u_1}{c_1} = \frac{500\ \frac{m}{s}}{347.2\ \frac{m}{s}} = 1.4401 \tag{5.173}$$

$$T_o = T_1\left(1 + \frac{\gamma-1}{2}\ M_1^2\right) \tag{5.174}$$

$$T_o = 300\ K\ \left(1 + \frac{1}{5}\ 1.4401^2\right) = 424.43\ K \tag{5.175}$$

$$P_o = P_1\left(1 + \frac{\gamma-1}{2}\ M_1^2\right)^{\frac{\gamma}{\gamma-1}} \tag{5.176}$$

$$P_o = 100\ kPa\ \left(1 + \frac{1}{5}\ 1.4401^2\right)^{3.5} = 336.828\ kPa \tag{5.177}$$

Now calculate the Prandtl-Meyer function for the freestream:

$$\nu(M_1) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_1^2-1\right)} - \tan^{-1}\sqrt{M_1^2-1} \tag{5.178}$$

$$\nu(M_1) = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\sqrt{\frac{1.4-1}{1.4+1}\left(1.4401^2-1\right)} - \tan^{-1}\sqrt{1.4401^2-1} \tag{5.179}$$

$$\nu(M_1) = 0.177138\ rad \tag{5.180}$$

$$\nu(M_1) = 0.177138\ rad\ \frac{180^\circ}{\pi\ rad} = 10.1493^\circ \tag{5.181}$$

The interpretation here is that an initially sonic flow would have had to had turned 10.1493° to achieve a Mach number of $M_1 = 1.4401$.

Now add on the actual turning:

$$\nu(M_2) = \nu(M_1) + 30^\circ \tag{5.182}$$

$$\nu(M_2) = 10.1493^\circ + 30^\circ = 40.1493^\circ \tag{5.183}$$

$$\nu(M_2) = 40.1493^\circ\ \frac{\pi\ rad}{180^\circ} = 0.700737\ rad \tag{5.184}$$

A trial and error solution gives the M_2 which corresponds to $\nu(M_2) = 0.700737\ rad$:

$$0.700737\ rad = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\sqrt{\frac{1.4-1}{1.4+1}\left(M_2^2-1\right)} - \tan^{-1}\sqrt{M_2^2-1} \tag{5.185}$$

$$M_2 = 2.54431 \tag{5.186}$$

$$T_2 = T_o\left(1 + \frac{\gamma-1}{2}\ M_2^2\right)^{-1} \tag{5.187}$$

$$T_2 = 424.43\ K\ \left(1 + \frac{1}{5}\ 2.54331^2\right)^{-1} = 189.4\ K \tag{5.188}$$

$$P_2 = P_o \left(1 + \frac{\gamma - 1}{2}\, M_2^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{5.189}$$

$$P_2 = 336.828\ kPa\ \left(1 + \frac{1}{5}\, 2.54331^2\right)^{-3.5} = 18.43\ kPa \tag{5.190}$$

$$\rho_2 = \frac{P_2}{RT_2} = \frac{18.43\ kPa}{\left(0.287\ \frac{J}{kg\ K}\right)(189.4\ K)} = 0.3390\ \frac{kg}{m^3} \tag{5.191}$$

$$c_2 = \sqrt{\gamma R T_2} = \sqrt{1.4\left(287\ \frac{J}{kg\ K}\right)(189.4\ K)} = 275.87\ \frac{m}{s} \tag{5.192}$$

$$w_2 = M_2 c_2 = 2.54431\left(275.87\ \frac{m}{s}\right) = 701.89\ \frac{m}{s} \tag{5.193}$$

5.6 Wave interactions and reflections

Shocks and rarefactions can intersect and reflect in a variety of ways.

5.6.1 Oblique shock reflected from a wall

An oblique shock which reflects from a wall is represented in Figure 5.9.

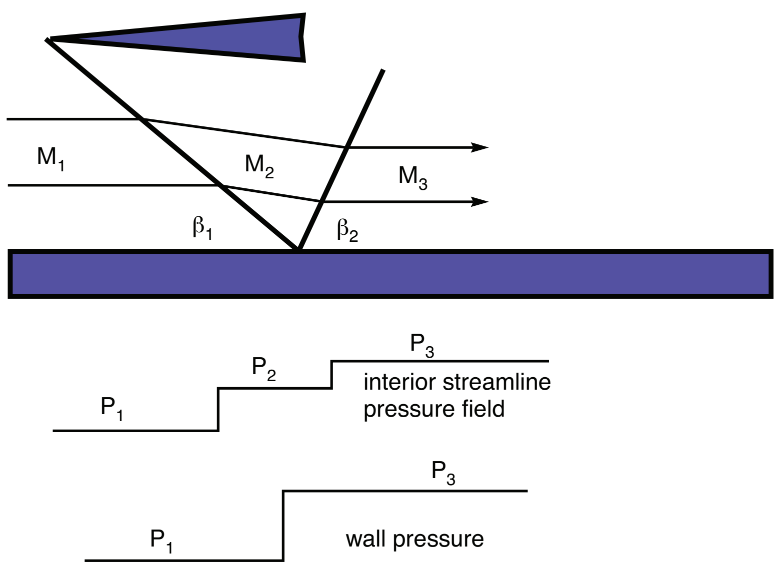

Figure 5.9: Reflection of an oblique shock from a wall

Note:

- analysis just that of two oblique shocks
- flow always turns to be parallel to wall
- angle of incidence *not* equal angle of reflection due to non-linear effects
- interior pressure profile has two steps
- wall pressure profile has single step
- $P_2 > 2P_1$, that is the pressure is *higher* than that obtained in the acoustic limit

5.6.2 Oblique shock intersection

Two oblique shocks intersect as sketched in Figure 5.10

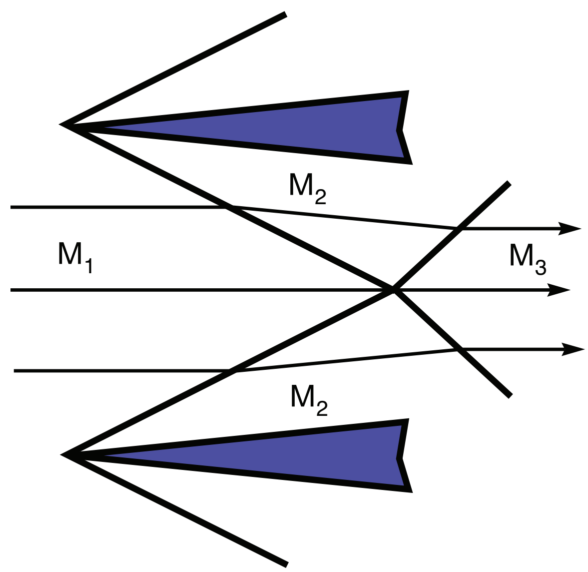

Figure 5.10: Interaction of two oblique shocks

Note:

- flow always turns to be parallel to wall
- when shocks intersect, flow turns again to be parallel to itself

5.6.3 Shock strengthening

A flow turned by a corner through an oblique shock can be strengthened by a second turn as sketched in Figure 5.11

Note: three new waves generated

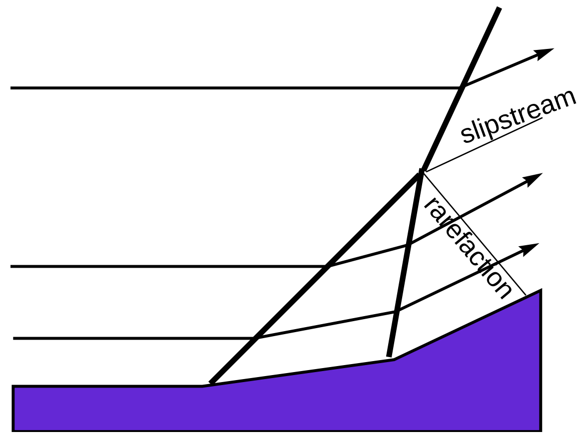

Figure 5.11: Shock Strengthening Sketch

- strengthened shock
- slipstream in which pressures match, velocity directions match, but velocity magnitudes differ
- weak rarefaction wave

5.6.4 Shock weakening

A flow turned by a corner through an oblique shock can be weakened by a second turn as sketched in Figure 5.12

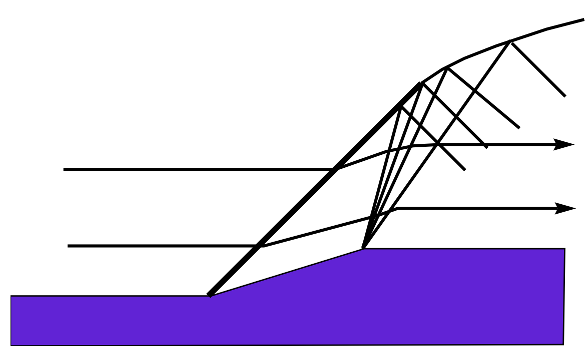

Figure 5.12: Shock Weakening Sketch

5.7 Supersonic flow over airfoils

The standard problem in flow over an airfoil is to determine the lift and the drag. While in actual design it is the magnitude of the lift force F_L, and drag force F_D, that is most crucial, there exists dimensionless numbers the lift coefficient C_L and the drag coefficient C_D which give good relative measures of airfoil performance.

$$C_L \equiv \frac{F_L}{\frac{1}{2}\rho_1 u_1^2 A} \tag{5.194}$$

$$C_D \equiv \frac{F_D}{\frac{1}{2}\rho_1 u_1^2 A} \tag{5.195}$$

Though this is the traditional formula, it is probably not the best for interpreting how the forces vary when flight speed is varied. This is because when u_1, flight speed is varied both numerator and denominator change. To remedy this, we can instead scale by the ambient sound speed to define a dimensionless lift force $F{*}_L$ and dimensionless drag force $F{*}_D$:

$$F{*}_L \equiv \frac{F_L}{\rho_1 c_1^2 A} \tag{5.196}$$

$$F{*}_D \equiv \frac{F_D}{\rho_1 c_1^2 A} \tag{5.197}$$

5.7.1 Flat plate at angle of attack

The simplest problem is that of a flat plate at angle of attack α_o. A schematic is illustrated in Figure 5.13. Note:

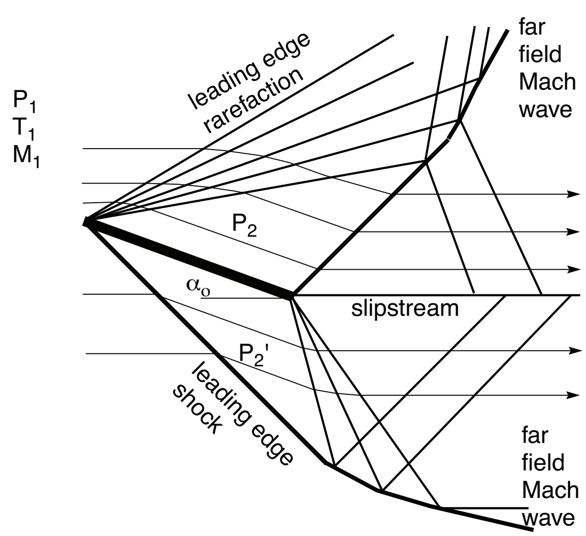

Figure 5.13: Supersonic Flow over a Flat Plate

- flow over the top is turned through an *isentropic* rarefaction to P_2

- flow over the bottom is turned through an oblique shock to P_2'
- Since $P_2' > P_2$, there is both lift and drag forces!
- both a shock and rarefaction are attached to the trailing edge to turn the flow to the horizontal
- the flow regions are separated by a slipstream in which pressure and velocity *directions* match
- $F_L = (P_2' - P_2)\, A \cos\alpha_o, C_L = \frac{\left(P_2'-P_2\right)\cos\alpha_o}{\frac{1}{2}\rho_1 u_1^2}$
- $F_D = (P_2' - P_2)\, A \sin\alpha_o, C_D = \frac{\left(P_2'-P_2\right)\sin\alpha_o}{\frac{1}{2}\rho_1 u_1^2}$
- the drag here is known as *wave drag* or *induced drag*
- other components of drag, *skin friction drag* and *thickness drag* are zero due to inviscid limit and zero thickness limit

In the small disturbance limit

$$\frac{\Delta P}{P_1} = \frac{\gamma M_1^2}{\sqrt{M_1^2 - 1}}\, \Delta\theta \tag{5.198}$$

$$P_2' = P_1 + \frac{\gamma P_1 M_1^2}{\sqrt{M_1^2 - 1}}\, \alpha_o \tag{5.199}$$

$$P_2 = P_1 + \frac{\gamma P_1 M_1^2}{\sqrt{M_1^2 - 1}}\, (-\alpha_o) \tag{5.200}$$

$$P_2' - P_2 = \frac{2\gamma P_1 M_1^2}{\sqrt{M_1^2 - 1}}\, \alpha_o \tag{5.201}$$

$$P_2' - P_2 = \frac{2\gamma P_1}{\sqrt{M_1^2 - 1}}\, \frac{u_1^2}{\frac{\gamma P_1}{\rho_1}}\, \alpha_o \tag{5.202}$$

$$P_2' - P_2 = \frac{2}{\sqrt{M_1^2 - 1}}\, \rho_1 u_1^2\, \alpha_o \tag{5.203}$$

$$F_L = \frac{2}{\sqrt{M_1^2 - 1}}\, \rho_1 u_1^2\, \alpha_o A \cos\alpha_o \tag{5.204}$$

$$F_L = \frac{2}{\sqrt{M_1^2 - 1}}\, \rho_1 u_1^2\, \alpha_o A(1) \tag{5.205}$$

$$C_L = \frac{4\alpha_o}{\sqrt{M_1^2 - 1}} \tag{5.206}$$

$$F_D = \frac{2}{\sqrt{M_1^2 - 1}} \, \rho_1 u_1^2 \; \alpha_o A \sin\alpha_o \tag{5.207}$$

$$F_D = \frac{2}{\sqrt{M_1^2 - 1}} \, \rho_1 u_1^2 \; \alpha_o^2 A \tag{5.208}$$

$$C_D = \frac{4 \; \alpha_o^2}{\sqrt{M_1^2 - 1}} \tag{5.209}$$

$$F*_L = \frac{F_L}{\rho_1 c_1^2 A} = \frac{2 \; M_1^2 \alpha_o}{\sqrt{M_1^2 - 1}} \tag{5.210}$$

$$F*_D = \frac{F_D}{\rho_1 c_1^2 A} = \frac{2 \; M_1^2 \alpha_o^2}{\sqrt{M_1^2 - 1}} \tag{5.211}$$

$$\text{High Mach number limit:} \qquad F*_L = 2 M_1 \alpha_o \tag{5.212}$$

$$\text{High Mach number limit:} \qquad F*_D = 2 M_1 \alpha_o^2 \tag{5.213}$$

Dimensionless lift and drag are plotted versus Mach number in Figure 5.14

Example 5.3

Lift and Drag on an Inclined Flat Plate

Given: Flat plate, of chord length 2 m, depth 10 m inclined at 20° to the horizontal in a freestream of $M_1 = 3$, $P_1 = 100 \; kPa$, $T_1 = 300 \; K$.

Find: Lift and drag forces on the plate.

Analysis: First some preliminaries:

$$c_1 = \sqrt{\gamma R T_1} = \sqrt{(1.4)\left(287 \; \frac{J}{kg \; K}\right)(300 \; K)} = 347.2 \; \frac{m}{s} \tag{5.214}$$

$$u_1 = M_1 c_1 = (3.0)\left(347.2 \; \frac{m}{s}\right) = 1,041.6 \; \frac{m}{s} \tag{5.215}$$

$$\rho_1 = \frac{P_1}{R T_1} = \frac{100,000 \; Pa}{\left(287 \; \frac{J}{kg \; K}\right)(300 \; K)} = 1.1614 \; \frac{kg}{m^3} \tag{5.216}$$

$$P_o = P_1 \left(1 + \frac{\gamma - 1}{2} \; M_1^2\right)^{\frac{\gamma}{\gamma-1}} \tag{5.217}$$

$$P_o = 100 \; kPa \; \left(1 + \frac{1}{5} \; 3^2\right)^{3.5} = 367.327 \; kPa \tag{5.218}$$

In a previous example, we found the oblique shock state under identical conditions:

$$P_2' = 377,072 \; Pa \tag{5.219}$$

Now consider the rarefaction.

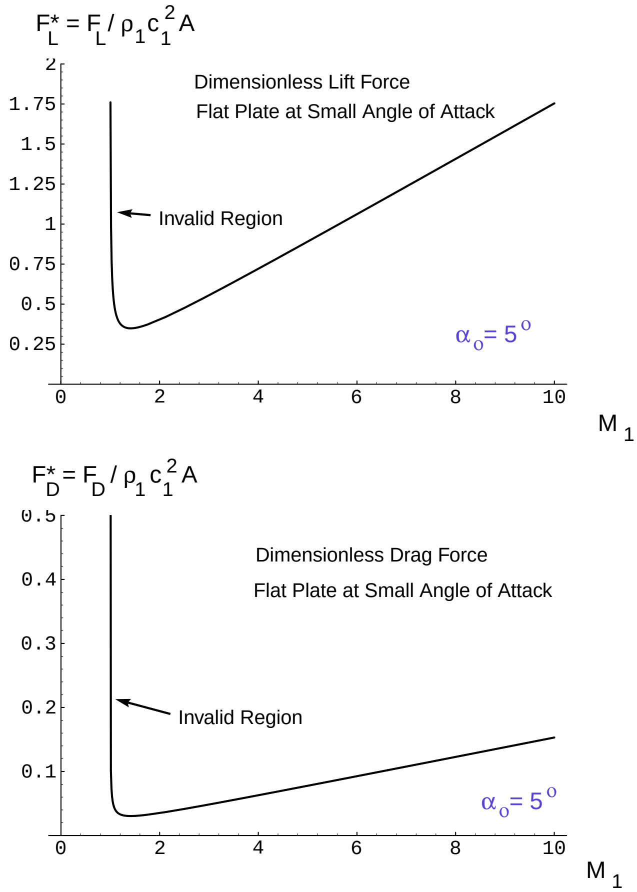

Figure 5.14: Dimensionless Lift and Drag versus Incoming Mach Number for Flat Plate at Small Angle of Attack

$$\nu(M_1) = \sqrt{\frac{\gamma+1}{\gamma-1}}\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}\left(M_1^2-1\right)} - \tan^{-1}\sqrt{M_1^2-1} \tag{5.220}$$

$$\nu(M_1) = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\sqrt{\frac{1.4-1}{1.4+1}\left(3^2-1\right)} - \tan^{-1}\sqrt{3^2-1} = 0.8691\ rad = 49.7973^\circ \tag{5.221}$$

$$\nu(M_2) = \nu(M_1) + 20^\circ \tag{5.222}$$

$$\nu(M_2) = 49.7973^\circ + 20^\circ = 69.7973^\circ \tag{5.223}$$

$$69.7973^\circ = 1.218\ rad = \sqrt{\frac{1.4+1}{1.4-1}}\tan^{-1}\sqrt{\frac{1.4-1}{1.4+1}\left(M_2^2-1\right)} - \tan^{-1}\sqrt{M_2^2-1} \tag{5.224}$$

$$M_2 = 4.3209 \tag{5.225}$$

$$P_2 = P_o\left(1 + \frac{\gamma-1}{2}\ M_2^2\right)^{-\frac{\gamma}{\gamma-1}} \tag{5.226}$$

$$P_2 = 367.327\ kPa\ \left(1 + \frac{1}{5}\ 4.3209^2\right)^{-3.5} = 1.591\ kPa \tag{5.227}$$

$$F_L = (P_2' - P_2)\, A \cos\alpha_o \qquad (5.228)$$

$$F_L = (377,072\ Pa - 1,591\ Pa)\,(10\ m)\,(2\ m)\cos 20^\circ \qquad (5.229)$$

$$F_L = 7,142,073\ N \qquad (5.230)$$

$$C_L = \frac{F_L}{\frac{1}{2}\rho_1 u_1^2 A} = \frac{7,142,073\ N}{\frac{1}{2}\left(1.1614\ \frac{kg}{m^3}\right)\left(1,041.6\ \frac{m}{s}\right)^2 (10\ m)\,(2\ m)} \qquad (5.231)$$

$$C_L = 0.5668 \qquad (5.232)$$

$$F_D = (P_2' - P_2)\, A \sin\alpha_o \qquad (5.233)$$

$$F_D = (377,072\ Pa - 1,591\ Pa)\,(10\ m)\,(2\ m)\sin 20^\circ \qquad (5.234)$$

$$F_D = 2,320,600\ N \qquad (5.235)$$

$$C_D = \frac{F_D}{\frac{1}{2}\rho_1 u_1^2 A} = \frac{2,320,600\ N}{\frac{1}{2}\left(1.1614\ \frac{kg}{m^3}\right)\left(1,041.6\ \frac{m}{s}\right)^2 (10\ m)\,(2\ m)} \qquad (5.236)$$

$$C_D = 0.1842 \qquad (5.237)$$

Compare with thin airfoil theory: $$C_{L\ thin} = \frac{4\alpha_o}{\sqrt{M_1^2 - 1}} \qquad (5.238)$$

$$C_{L\ thin} = \frac{4\,(20^\circ)\,\frac{\pi\ rad}{180^\circ}}{\sqrt{3^2 - 1}} = 0.4936 \qquad (5.239)$$

$$C_{D\ thin} = \frac{4\alpha_o^2}{\sqrt{M_1^2 - 1}} \qquad (5.240)$$

$$C_{D\ thin} = \frac{4\left((20^\circ)\,\frac{\pi\ rad}{180^\circ}\right)^2}{\sqrt{3^2 - 1}} = 0.1723 \qquad (5.241)$$

5.7.2 Diamond-shaped airfoil

The simplest supersonic airfoil with camber for analysis purposes is the diamond shaped airfoil as sketched in Figure 5.15. The sketch shows the airfoil at zero angle of attack. The upper half of the wedge is inclined at angle ϵ to the horizontal In this case there will be no lift but there will be drag. Note the following features:

- sudden turn through lead oblique shock
- turn through isentropic Prandtl-Meyer rarefaction
- final turn through oblique shock attached to trailing edge
- far field limit: acoustic (Mach) waves
- Thin airfoil limit $C_{L\ thin} = \frac{4\epsilon}{\sqrt{M_1^2-1}}$, same as for flat plate!
- Thin airfoil limit $C_{D\ thin} = \frac{4\epsilon^2}{\sqrt{M_1^2-1}}$, same as for flat plate!

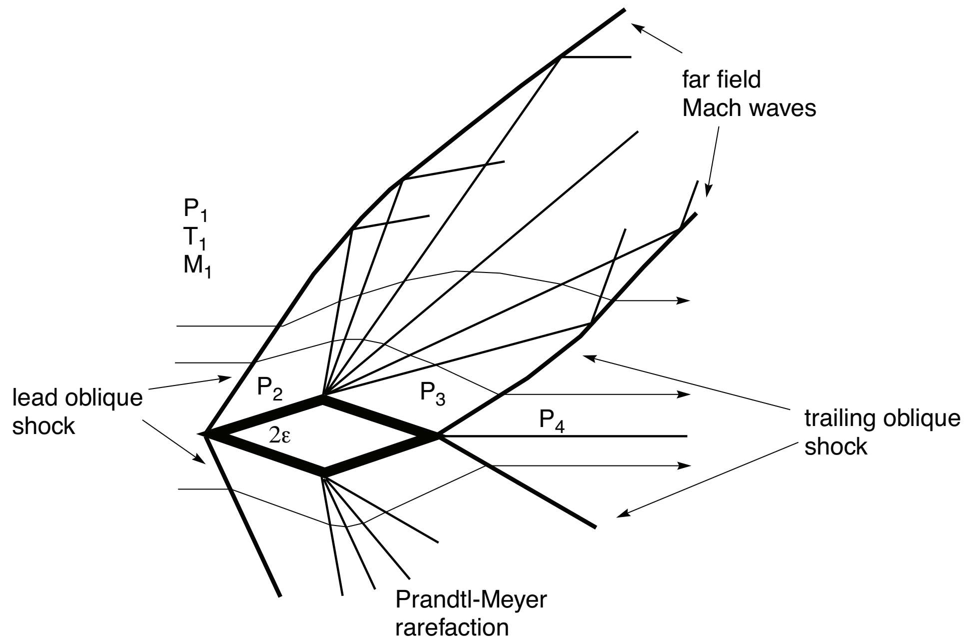

Figure 5.15: Supersonic Flow over a Diamond-Shaped Airfoil

5.7.3 General curved airfoil

A general airfoil with camber is sketched in Figure 5.16. The sketch shows the airfoil at zero angle of attack. In this case there will be no lift but there will be drag. Note the following features:

- lead oblique shock
- lead shock weakened by series of non-centered rarefaction waves
- shock at trailing edge, also weakened by non-centered rarefaction waves
- far field: acoustic (Mach) waves

5.7.4 Transonic transition

Transonic flow exists whenever there is a continuous transition from subsonic to supersonic flow. One example of a transonic flow is sketched in Figure 5.17[1] which shows an accelerating airfoil.

Note:

[1]adopted from Bryson, A. E., "An Experimental Investigation of Transonic Flow Past Two-Dimensional Wedge and Circular-Arc Sections Using a Mach-Zehnder Interferometer," *NACA Tech. Note 2560*, 1951.

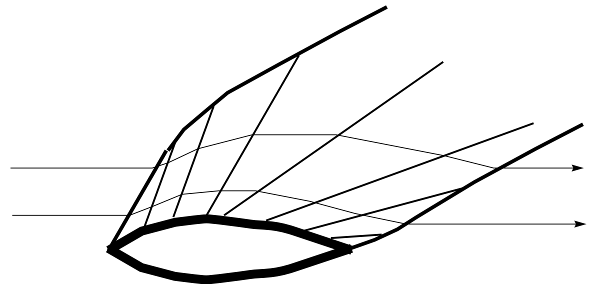

Figure 5.16: Supersonic Flow over a Curved Airfoil

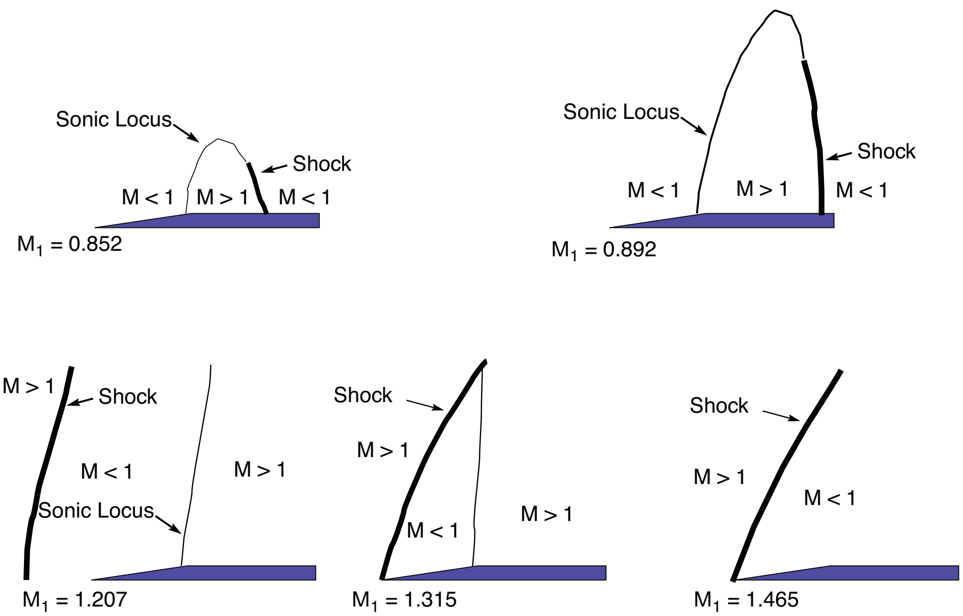

Figure 5.17: Transition from Subsonic to Transonic to Supersonic Flow

- for high subsonic Mach number a bubble of supersonic flow appears
- smooth transition from subsonic to supersonic
- shock transition from supersonic to subsonic
- as Mach number increases, supersonic bubble expands
- for slightly supersonic Mach number, new shock approaches from far field
- as supersonic Mach number increases, shock from far field approaches leading edge and supersonic bubble disappears
- challenging problems, not easily solved till 1960's!

Chapter 6

Linear flow analysis

see Anderson, Chapter 9

In this section we consider flows which are

- steady,
- two-dimensional,
- irrotational,
- isentropic,
- calorically perfect, and
- ideal.

The analysis is extensible to other cases.

6.1 Formulation

In lecture a detailed discussion is given in which the linearized velocity potential equation is obtained:

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{6.1}$$

6.2 Subsonic flow

Here we consider flows in which the Mach number is subsonic, but not negligibly small.

6.2.1 Prandtl-Glauret rule

A discussion is given where it is shown that the pressure coefficient on a supersonic airfoil can be determined in terms of the pressure coefficient known from subsonic theory:

$$c_p = \frac{c_{po}}{\sqrt{1 - M_\infty^2}}. \tag{6.2}$$

6.2.2 Flow over wavy wall

The technique of separation of variables is used to show the subsonic flow over a wavy wall can be written in terms of the velocity potential as

$$\phi(x, y) = \frac{U_\infty h}{\sqrt{1 - M_\infty^2}} \sin\left(\frac{2\pi x}{l}\right) \exp\left(\frac{-2\pi\sqrt{1 - M_\infty^2}\, y}{l}\right). \tag{6.3}$$

6.3 Supersonic flow

6.3.1 D'Alembert's solution

The D'Alembert solution for the wave equation is shown for supersonic flows:

$$\phi(x, y) = f\left(x + \sqrt{M_\infty^2 - 1}\, y\right) + g\left(x - \sqrt{M_\infty^2 - 1}\, y\right). \tag{6.4}$$

6.3.2 Flow over wavy wall

The solution for flow over a wavy wall is given in detail in lecture.

Chapter 7

Viscous flow

This chapter will focus on problems in which viscous stress plays an important role in determining the motion of the fluid. The topic in general is quite broad; to gain understanding of the fundamental physics, we will restrict our attention to the following limits:

- incompressible fluid
- isotropic Newtonian fluid with constant properties
- at most two-dimensional unsteady flow

The chapter will consider the governing equations and then solve a few representative problems.

7.1 Governing equations

This section considers the governing equations for the conditions specified for this chapter. In dimensional non-conservative form, the governing equations are as follows:

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0\\
\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} &= -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\\
\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} &= -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\\
\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) &= \left(\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial y}\right) + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\\
&\quad +2\mu\left(\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right)
\end{aligned}
$$

An argument could be made to eliminate the viscous dissipation term and the pressure derivatives in the energy equation. The argument is subtle and based on the low Mach number limit which corresponds to incompressibility.

7.2 Couette flow

Consider a channel flow driven by plate motion See Figure 7.1

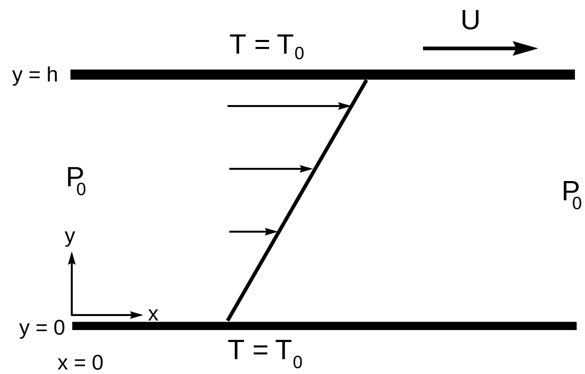

Figure 7.1: Sketch for Couette flow

The mechanics of such a flow can be described by stripping away many extraneous terms from the governing equations.

Take

- fully developed velocity and temperature profiles: $\frac{\partial u}{\partial x} \equiv 0$, $\frac{\partial T}{\partial x} \equiv 0$
- steady flow $\frac{\partial}{\partial t} \equiv 0$
- constant pressure field $P(x, y, t) = P_o$
- constant temperature channel walls $T(x, 0, t) = T(x, h, t) = T_o$

Since fully developed mass gives:

$$\frac{\partial v}{\partial y} = 0 \tag{7.1}$$

$$v(x, y) = f(x) \tag{7.2}$$

and since in order to prevent mass flowing through the wall boundaries, $v(x, 0) = v(x, h) = 0$, thus

$$v(x, y) = 0 \tag{7.3}$$

Since $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial t} = 0$ and $\frac{\partial T}{\partial x} = 0$, $\frac{\partial T}{\partial t} = 0$, we have at most,

$$u = u(y) \tag{7.4}$$
$$T = T(y) \tag{7.5}$$

The y momentum equation has no information and x momentum and energy reduce to the following:

$$0 = \mu \frac{d^2 u}{dy^2} \tag{7.6}$$
$$0 = k \frac{d^2 T}{dy^2} + \mu \left(\frac{du}{dy} \right)^2 \tag{7.7}$$

The x momentum equation is thus

$$\frac{d^2 u}{dy^2} = 0 \tag{7.8}$$
$$\frac{du}{dy} = C_1 \tag{7.9}$$
$$u(y) = C_1 y + C_2 \tag{7.10}$$

Now applying $u(0) = 0$ and $u(h) = U$ to fix C_1 and C_2 we get

$$u(y) = U \frac{y}{h} \tag{7.11}$$

Shear stress:

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} \tag{7.12}$$
$$\tau_{yx} = \mu \frac{U}{h} \tag{7.13}$$

The energy equation becomes

$$\frac{d^2 T}{dy^2} = -\frac{\mu}{k} \left(\frac{du}{dy} \right)^2 \tag{7.14}$$

$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\frac{U^2}{h^2} \tag{7.15}$$

$$\frac{dT}{dy} = -\frac{\mu}{k}\frac{U^2}{h^2}\ y + C_1 \tag{7.16}$$

$$T(y) = -\frac{1}{2}\frac{\mu}{k}\frac{U^2}{h^2}\ y^2 + C_1 y + C_2 \tag{7.17}$$

Now $T(0) = T_o$ and $T(h) = T_o$. This fixes the constants, so

$$T(y) = \frac{1}{2}\frac{\mu}{k}U^2\left(\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right) + T_o \tag{7.18}$$

In dimensionless form this becomes

$$\frac{T - T_o}{T_o} = \frac{1}{2}\frac{\mu c_p}{k}\frac{U^2}{c_p T_o}\left(\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right) \tag{7.19}$$

$$\frac{T - T_o}{T_o} = \frac{Pr\ Ec}{2}\left(\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right) \tag{7.20}$$

$$\text{Prandtl Number:} \qquad Pr \equiv \frac{\mu c_p}{k} = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho c_p}} = \frac{\nu}{\alpha} = \frac{momentum\ diffusivity}{thermal\ diffusivity} \tag{7.21}$$

$$\text{Eckert Number:} \qquad Ec \equiv \frac{U^2}{c_p T_o} = \frac{kinetic\ energy}{thermal\ energy} \tag{7.22}$$

Now

$$\frac{dT}{dy} = \frac{1}{2}\frac{\mu}{k}\frac{U^2}{h^2}(h - 2y) \tag{7.23}$$

$$q_y = -k\frac{dT}{dy} = \frac{1}{2}\mu\frac{U^2}{h^2}(2y - h) \tag{7.24}$$

$$q_y(0) = -\frac{\mu U^2}{2h} \tag{7.25}$$

Note:

- at lower wall, heat flux into wall; heat generated in fluid conducted to wall
- wall heat flux magnitude independent of thermal conductivity
- higher plate velocity, higher wall heat flux
- higher viscosity, higher wall heat flux
- thinner gap, higher wall heat flux

Also since T_{max} occurs at $y = \frac{h}{2}$

$$T_{max} = \frac{1}{8}\frac{\mu}{k}U^2 + T_o \tag{7.26}$$

Note:

- high viscosity, high maximum temperature
- high plate velocity, high maximum temperature
- low thermal conductivity, high maximum temperature

Dimensionless wall heat flux given by the Nusselt number:

$$Nu \equiv \left| \frac{q_y(0)}{\frac{k\Delta T}{\Delta y}} \right| = \left| \frac{q_y(0)\Delta y}{k\Delta T} \right| \tag{7.27}$$

$$Nu = \frac{\frac{\mu U^2}{2h}\frac{h}{2}}{k\frac{1}{8}\frac{\mu}{k}U^2} = 2 \tag{7.28}$$

7.3 Suddenly accelerated flat plate

The problem of pulling a plate suddenly in a fluid which is initially at rest is often known as Stokes' First Problem or Rayleigh's problem.

7.3.1 Formulation

Consider a channel flow driven by a suddenly accelerated plate. See Figure 7.2 Initially, $t < 0$

- fluid at rest
- plate at rest

For $t \geq 0$

- plate pulled at constant velocity U

Assume:

- constant pressure $P(x, y, t) = P_o$
- fully developed flow $\frac{\partial u}{\partial x} = 0$, $\frac{\partial T}{\partial x} = 0$

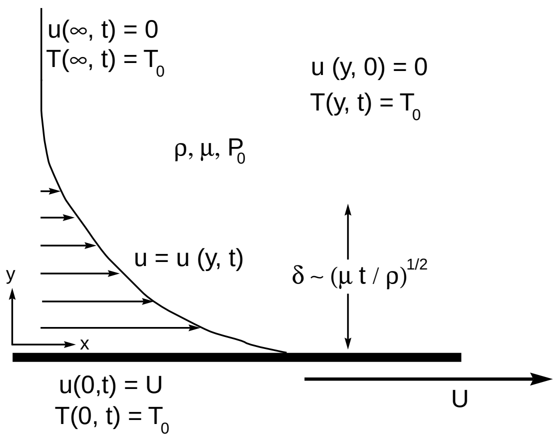

Figure 7.2: Sketch for Stokes' First Problem

Again from mass we deduce that $v(x, y, t) = 0$. The x momentum equation reduces to

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \tag{7.29}$$

The initial and boundary conditions are

$$u(y, 0) = 0 \tag{7.30}$$
$$u(0, t) = U \tag{7.31}$$
$$u(\infty, t) = 0 \tag{7.32}$$

7.3.2 Velocity profile

This problem is solved in detail in lecture. The solution for the velocity field is shown to be

$$\frac{u}{U_o} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{\nu t}} \exp\left(-s^2\right)\ ds \tag{7.33}$$

7.4 Starting transient for plane Couette flow

The starting transient problem for plane Couette flow can be formulated as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{7.34}$$
$$u(y, 0) = 0, \qquad u(0, t) = U_o, \qquad u(h, t) = 0. \tag{7.35}$$

In class a detailed solution is presented via the technique of separation of variables. The solution is

$$\frac{u}{U_o} = 1 - \frac{y}{h} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(\frac{-n^2\pi^2\nu t}{h^2}\right)\sin\left(\frac{n\pi y}{h}\right) \tag{7.36}$$

7.5 Blasius boundary layer

The problem of flow over a flat plate in the absence of pressure gradient is formulated and solved using the classical approach of Blasius.

7.5.1 Formulation

After suitable scaling and definition of similarity variables, discussed in detail in class, the following third order non-linear ordinary differential equation is obtained:

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0, \tag{7.37}$$

$$\left.\frac{df}{d\eta}\right|_{\eta=0} = 0, \qquad \left.\frac{df}{d\eta}\right|_{\eta\to\infty} = 1, \qquad f|_{\eta=0} = 0. \tag{7.38}$$

This equation is solved numerically as a homework problem.

7.5.2 Wall shear stress

The solution is used to obtain the classical formulae for skin friction coefficient:

$$C_f = \frac{0.664}{\sqrt{Re_x}}, \tag{7.39}$$

and drag coefficient:

$$C_D = \frac{1.328}{\sqrt{Re_L}}. \tag{7.40}$$

Chapter 8

Acoustics

This chapter outlines the brief introduction to acoustics given in class in somewhat more detail.

8.1 Formulation

We reduce the Euler equations for isentropic flow to the following equations where quantities with a hat are understood to be small perturbations about the ambient state, denoted with a subscript of ”o”.

$$\frac{\partial \hat{\rho}}{\partial t} + \rho_o \nabla \cdot \hat{\mathbf{v}} = 0 \tag{8.1}$$

$$\rho_o \frac{\partial \hat{\mathbf{v}}}{\partial t} + \nabla \cdot \hat{P} = 0 \tag{8.2}$$

$$\hat{P} = c_o^2 \hat{\rho}. \tag{8.3}$$

Introducing the velocity potential $\nabla \phi = \hat{\mathbf{v}}$ and employing further manipulation allows the equation to be written as the wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = c_o^2 \nabla^2 \phi. \tag{8.4}$$

The pressure, velocity, and density are then obtained from

$$\hat{P} = -\rho_o \frac{\partial \phi}{\partial t}, \tag{8.5}$$

$$\hat{\mathbf{v}} = \nabla \phi, \tag{8.6}$$

$$\hat{\rho} = -\rho_o c_o^2 \frac{\partial \phi}{\partial t}. \tag{8.7}$$

8.2 Planar waves

The D'Alembert solution for planar waves is shown in class to be

$$\phi = f(x + c_o t) + g(x - c_o t), \tag{8.8}$$

$$\hat{P} = -\rho_o c_o f'(x + c_o t) + \rho_o c_o g'(x - c_o t), \tag{8.9}$$

$$\hat{u} = f'(x + c_o t) + g'(x - c_o t), \tag{8.10}$$

$$\tag{8.11}$$

8.3 Spherical waves

The D'Alembert solution for spherical waves is shown in class to be

$$\phi = \frac{1}{r} f(r + c_o t) + \frac{1}{r} g(r - c_o t), \tag{8.12}$$

$$\hat{P} = -\frac{\rho_o c_o}{r} f'(r + c_o t) + \frac{\rho_o c_o}{r} g'(r - c_o t), \tag{8.13}$$

$$\hat{u} = \frac{1}{r} f'(r + c_o t) + \frac{1}{r} g'(r - c_o t), \tag{8.14}$$

$$\tag{8.15}$$