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A BEHAVIORAL SUMMARY FOR COMPLETELY RANDOM NETS

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This paper characterizes the cycle structure of a completely random net. Variables such as number of cycles of a specified length, number of cycles, number of cyclic states and length of cycle are studied. A square array of indicator variables enables convenient study of moment structure. Additionally, exact and asymptotic distributional results are presented.

1. *Introduction.* Random switching net models have been employed in various contexts to aid in understanding and simulating aspects of the behavior of biological systems. Two of the most notable examples are neural net models and genetic net models. The former model components of the central nervous system and are typically developed from formal, in the sense of McCulloch and Pitts (1943), neurons. Discussions are given in Griffith (1971) and in Arbib (1972). The latter model the genetic structure of cells and have been presented in a series of papers by Kauffman (1969a, 1969b, 1970, 1971, 1974). Recently, Cavender (1977) has noted that switching net models can provide behavioral agreement with that of a wide range of biological organisms.

As Sherlock (1979) notes, analytical limitations rather than lack of biological relevance have generally tended to hinder the impact of this type of modelling enterprise. However, for one instance, that of a completely random switching net, virtually complete mathematical description can be provided. While completely random networks are not in themselves benchmarks which demonstrate the need to exercise structural and/or functional control over the net models. That is, from knowledge of the behavior of very large completely random switching nets, we learn how to formulate restricted net models exhibiting more (real world) plausible behavior. The utilization of "threshold levels" for neurons in a neural net or of "forcing inputs" to genes in a genetic net are illustrations of the imposition of such control. The intent of this article is to present a concise, accessible behavioral summary of the completely random situation. While some of the results in the sequel have been discussed by others, the primary objectives here are to unify, clarify and simplify.

A switching net consists of a set of N elements having an associated interconnectance structure. Each element has binary response to input information and at any instant in time the net is in one of 2^N distinct states. The net operates on discrete (clock) time such that its state at time t uniquely determines its state at time $t+1$. For a completely random net the choice of successor state to a state is made via an equally likely selection from all 2^N possible net states.

Since the number of net states is finite and the net is deterministic, given an arbitrary initial state, the sequence of states arising from this initial one must necessarily encounter a state it has previously been in. Thereafter, it must repeat this intermediate sequence of states. Such a sequence of states is called a cycle. The number of distinct states in the cycle is called the cycle length. For a given net, some (at least one) elements will be cyclic but others may be transient occurring during a run-in prior to cycling. For a given net there may be an assortment of cycles of varying lengths. Thus, each net creates a cycle structure (or space) which, for small N, is typically depicted through a state diagram.

Any net may be described in terms of its transition matrix. That is, placing net states in one-to-one correspondence with the integers from 1 through 2^N we can form a $2^N \times 2^N$ matrix T, whose entries T_{ij} are such that

$$T_{ij} = \begin{cases} 1 & \text{if state } i \text{ is successor to state } j \\ 0 & \text{if otherwise.} \end{cases}$$

The transition matrix representation of a net shows that any net may be viewed as a transformation from a set of $n = 2^N$ elements into itself. This description suffices for examination of the cycle structure of a net but sacrifices the binary character of the net. However, in the completely random net the binary aspect is immaterial and thus in the sequel, we study the mathematically equivalent form of the problem, the characterization of the cycle space of a random transformation on a finite set. Griffith recognizes this equivalence (sec. 8.2.3.) in offering several behavorial results. Kauffman does as well (1970), in discussing the results of simulations of completely random nets. Amongst the important mathematical efforts on this problem are the papers of Rubin and Sitgreaves (1954), Harris (1960), Katz (1955), Kruskal (1954), Folkert (1955), Cull (1971, 1978) and Gontcharoff (1944). Chapter 4 of the book by Riordan (1958) on cycles of permutations is directly relevant. As noted earlier, the reader may find some of the ensuing results across these references.

In concluding this section, we return to the transition matrix T. By definition, T has exactly one "1" per column. Suppose T results in a cycle structure with k transient elements and m cycles of length $r_1, r_2, r_3, \ldots, r_m$ respectively. Then Cull (1978) shows that the characteristic polynomial of T, $|T-\lambda I|$, where operations are performed in the real field, will have the form

$$\pm\lambda^k \prod_{i=1}^{m} (\lambda^{r_i}-1).$$

Clearly, $Tr(T)$ gives the number of elements on cycles of length 1 and more generally, $Tr(T^m)$ yields the number of states on cycles whose length divides m. Hence, $Tr(T^{n!})$ equals the number of states on cycles and $n - Tr(T^{n!})$ equals the number of transient states. How may we obtain a matrix H_m from T such that $Tr(H_m)$ equals the number of states on cycles whose length is exactly m? Let $C_m = \{\text{primes} \leq m$ which appear in the prime representation of $m\}$ (i.e. appear with a power ≥ 1) and let N_m = number of elements in C_m. The number of subsets of N_m is 2^{N_m} and the number of subsets of size k is $\binom{N_m}{k} \equiv N_{mk}$. At a given k let j index the subsets of size k so that the 2^{N_m} subsets may be denoted by C_{kj}, $k = 0,1,2,\ldots, N_m$, $j = 1,2,\ldots, N_{mk}$. Let g_{kj} equal m divided by the product of all the elements in C_{kj}. Then

THEOREM. *For each m, $m = 1,2,\ldots, n$, let*

$$H_m = \sum_{k=0}^{N_m} (-1)^k \sum_{j=1}^{N_{mk}} T^{g_{kj}}.$$

Then $Tr(H_m)$ = number of states on cycles whose length is exactly m.

Proof. A direct inclusion–exclusion argument.

2. *Random Transformations and Cycle Structure Random Variables.* The selection of a random (equally likely) transformation T is conveniently accomplished as a sequence of n independent multinomial trials where the j^{th} trial chooses the successor to state j in an equiprobable fashion from amongst the n elements. Then, with respect to cycles of length l, $Tr(T)$ is distributed binomially $(n, (1/n))$ with limiting distribution Poisson (1). But what about longer cycles or, more generally, the nature of the cycle structure of such a random transformation? We will examine the distributions of:

(i) the number of cycles of a specified length
(ii) the number of cycles
(iii) the number of cyclic states
(iv) the length of a cycle.

These issues are not easily pursued through the matrices T^m and H_m.

We may readily study the moments of the random variables in (i)–(iii) through the $n \times n$ array of random variables.

$$\begin{array}{ccc} D_{11}^n & \cdots\cdots & D_{1n}^n \\ D_{21}^n & \cdots\cdots & D_{2n}^n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ D_{n1}^n & \cdots\cdots & D_{nn}^n \end{array}, \tag{1}$$

where

$$D_{ri}^n = \begin{cases} 1 & \text{if state } x_i \text{ is on a cycle of length } r \\ 0 & \text{if otherwise.} \end{cases}$$

Let

$$A_{n,r} = \sum_{i=1}^{n} D_{ri}^n = \text{number of states on a cycle of length } r, \tag{2}$$

$$B_{n,r} = r^{-1} A_{n,r} = \text{number of cycles of length } r, \tag{3}$$

$$C_i^n = \sum_{r=1}^{n} D_{ri}^n = \begin{cases} 1 & \text{if state } x_i \text{ is cyclic} \\ 0 & \text{if transient,} \end{cases} \tag{4}$$

$$U_n = \sum_{i=1}^{n} A_{n,r} = \sum_{i=1}^{n} C_i^n = \text{number of cyclic states} \tag{5}$$

and

$$V_n = \sum_{r=1}^{n} B_{n,r} = \text{number of cycles.} \tag{6}$$

For any fixed r the joint distribution of $D_{r1}^n, \ldots, D_{rn}^n$ or of any subset will be that of a collection of dependent interchangeable random variables. The marginal distribution of any D_{ri}^n is directly

$$P(D_{ri}^n = 1) = \binom{n-1}{r-1}(r-1)!\, n^{-r} = n^{-(r+1)}(n)_r, \tag{7}$$

where $(n)_r$ is the falling factorial of r terms starting at n. Immediately, then (with $\rightarrow$ indicating $n \rightarrow \infty$)

$$E(A_{n,r}) = n^{-r}(n)_r \rightarrow 1, \tag{8}$$

$$E(B_{n,r}) = r^{-1}n^{-r}(n)_r \rightarrow r^{-1}, \tag{9}$$

$$E(C_i^n) = n^{-1} \sum_{r=1}^{n} n^{-r}(n)_r \rightarrow 0, \tag{10}$$

$$E(U_n) = \sum_{r=1}^{n} n^{-r}(n)_r \rightarrow \infty \tag{11}$$

$$E(V_n) = \sum_{r=1}^{n} r^{-1}n^{-r}(n)_r \rightarrow \infty. \tag{12}$$

The limits imply that with increasing number of elements the probability of any particular element being cyclic tends to 0, but the expected number of cyclic states and of cycles tends to ∞.

To get moments of order m, we need the joint distribution of any subset of size m of the D_{ri}^n. For any pair, D_{ri}^n, D_{sj}^n, we have three cases: (i) $r \neq s$, $i \neq j$; (ii) $r = s$, $i \neq j$; and (iii) $r \neq s$, $i = j$. For (i) we have

$$P(D_{ri}^n = 1, D_{sj}^n = 1) = \begin{cases} n^{-(r+s+1)}(n-1)^{-1}(n)_{r+s}, & r+s \leq n; \\ 0, & r+s > n; \end{cases}$$

for (ii)

$$P(D_{ri}^n = 1, D_{rj}^n = 1) = \begin{cases} n^{-(r+1)}(n-1)^{-1}(r-1)(n)_r + n^{-(2r+1)}(n-1)^{-1}(n)_{2r}, \\ \quad r \leq n/2 \\ n^{-r+1}(n-1)^{-1}(r-1)(n)_r; \quad n/2 < r \leq n; \\ 0 \quad\quad r > n \end{cases}$$

for (iii)

$$P(D_{ri}^n = 1) = 0.$$

In each case, using the marginal, (7), we may complete the joint distribution. Thus, covariances between any pair of D_{ri}^n may be obtained and variances and covariances for the variables in (2)–(6) as well. In particular, $A_{n,r}$ and $A_{n,s}$ (also $B_{n,r}$ and $B_{n,s}$) are negatively correlated but asymptotically uncorrelated. Also, as $n \rightarrow \infty$ var $(A_{n,r}) \rightarrow r$, var $(B_{n,r}) \rightarrow r^{-1}$ and var $(C_i^n) \rightarrow 0$. Var $(U_n) \rightarrow \infty$ and var $(V_n) \rightarrow \infty$, but these are most easily shown using results in Section 4.

As to the joint distribution of any subset of size m of the D_{ri}^n, suppose first that all m variables are in the same row of (1). Taking $mr \leq n$, and using the interchangeability of the variables, we require

$$\begin{aligned} P_{n,m,r} &\equiv P(\text{states } \alpha_1, \ldots, \alpha_m, \text{ each on a cycle of length } r) \\ &= P(D^n_{r\alpha_1} = 1, D^n_{r\alpha_2} = 1, \ldots, D^n_{r\alpha_n} = 1. \end{aligned}$$

Consider all possible partitions of m with no part greater than r. If a given partition has parts $m_1, \ldots, m_j$, let $n(m_1, \ldots, m_j)$ be the number of ways to allocate m distinct objects into j like cells with m_i in cell i $(\Sigma_{i=1}^{j} m_i = m)$. Also associate with $m_1, m_2, \ldots, m_j$ the event $E_{nr}(m_1, \ldots, m_j)$ defined by {states $\alpha_1, \ldots, \alpha_{m_1}$ on the same cycle of length r, states $\alpha_{m_1+1}, \ldots, \alpha_{m_1+m_2}$ on the same cycle of length r, etc.}. If $\mathscr{S}_m$ is the set of all partitions of m, and $\mathscr{S}_{m,r}$ is the set of all partitions of m with no part greater than r, then

$$P_{n,m,r} = \sum_{\mathscr{S}_{m,r}} n(m_1, \ldots, m_j) P[E_{nr}(m_1, \ldots, m_j)],$$

where

$$P(E_{nr(m_1, \ldots, m_j)}) = [(n)_m]^{-1} (n)_{jr} n^{-jr} [(r-1)!]^j \left[\prod_{i=1}^{j} (r - m_i)! \right]^{-1}.$$

Using $P_{n,m-1,r}$, we may complete the joint distribution of the $mD^n_{r\alpha_i}$. Suppose that the m D^n_{ri} are all in the same column of (1), say $D^n_{\alpha_1 i}, D^n_{\alpha_2 i}, \ldots, D^n_{\alpha_m i}$. Then their joint distribution will be multinomial with associated $P_{\alpha_j} = n^{-(\alpha_j+1)}(n)_{\alpha_j}$, $j = 1,2, \ldots, n$. Combining the above ideas, we may obtain the joint distribution of any subset of size m of the D_{ri}. We omit the details.

3. *Exact Distributions.* The exact distribution of U_n can be obtained following ideas given by Rubin and Sitgreaves (1954). Given T, for any state x, let $S(x)$ be the set of all successors to x, i.e.

$$S(x) = \{x' \colon T^r x = x' \text{ for some } r \geq 0\}.$$

Then, with $k \geq +1$,
P(x has k successors, $S(x)$ has cycle of length r, x is not cyclic) $=$ $P(Tx \neq x;\ T^2x \neq Tx,\ T^2x \neq x;\ T^3x \neq T^2x,\ T^3x \neq Tx,\ T^3x \neq x;\ T^{k-1}x \neq T^{k-2}x, \ldots, T^{k-1}x \neq x;\ T^k x = T^{k-r}x)$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) \cdots \left(\frac{n-(k-1)}{n}\right)\left(\frac{1}{n}\right) = n^{-(k+1)}(n)_k.$$

Thus,

$$P(S(x) \text{ has cycle of length } r,\ x \text{ is not cyclic})$$
$$= \sum_{k=r+1}^{n} n^{-(k+1)}(n)_k. \tag{13}$$

But

$$\begin{aligned}
&P(S(x) \text{ has cycle of length } r,\ x \text{ is not cyclic})\\
&= \sum_{u=r}^{n} P(S(x) \text{ has cycle of length } r,\ x \text{ is not cyclic},\ U_n = u)\\
&= \sum_{u=r}^{n} P(S(x) \text{ has cycle of length } r | x \text{ is not cyclic},\ U_n = u)\ P(x \text{ is not cyclic}\\
&| U_n = u)\ P(U_n = u)
\end{aligned} \tag{14}$$

$$= \sum_{u=r}^{n} \left[1 \left(\frac{u-1}{u}\right)\left(\frac{u-2}{u-1}\right) \cdots \left(\frac{u-(r-1)}{u-(r-s)}\right)\left(\frac{1}{u-(r-1)}\right)\right] \left(\frac{n-u}{n}\right) P(U_n = u)$$

$$= \sum_{u=r}^{n} \frac{n-u}{nu} P(U_n = u).$$

Since (13) and (14) are equal for all r,

$$\begin{aligned}
&\sum_{k=r+1}^{n} n^{-(k+1)}(n)_k - \sum_{k=r+2}^{n} n^{-(k+1)}(n)_k,\\
&= \sum_{u=r}^{n} (nu)^{-1}(n-u)P(U_n = u) - \sum_{u=r\neq 1}^{n} (nu)^{-1}(n-u)P(U_n = u),
\end{aligned}$$

whence

$$P(U_n = u) = n^{-(u+1)}(n)_u u, \qquad u = 1,2,\ldots,n. \tag{15}$$

From (15), $P(U_n = n) = n^{-n}n!$. This is seen directly by noting that $U_n = n$ iff T is $l-1$ and that there are $n!$ such T. Harris (1960) offers an alternative development of (15) by decomposing the cycle space of T and employing a convenient identity from Katz (1955). Using (15), we have the identity

$$\sum_{u=1}^{n} n^{-u}(n)_u u = n.$$

Using (11) and (15), we have

$$E(U_n) = nE(U_n)^{-1}.$$

Similarly, we can develop identities for higher moments of U_n, e.g.

$$E(U_n^2) = 2n - E(U_n).$$

The exact distributions of V_n and $B_{n,r}$ may be obtained through their conditional distributions, given U_n. Such events as v cycles resulting from u cyclic states or k cycles of length r resulting from u cyclic states may be viewed in terms of cycle classes of permutations of u distinct objects. Riordan (1958), Chap. 4, is helpful here, e.g. from pp. 70–72 we have

$$\alpha(u, v) \equiv P(V_n = v \mid U_n = u) = (-l)^{u+v} s(u, v)(u!)^{-1}, \tag{16}$$

where $s(u, v)$ are Stirling numbers of the first kind. From the familiar recurrence relationship for such numbers (Riordan, 1958, p. 33), we obtain

$$\alpha(u, v) = u^{-1}(u-1)\alpha(u-1, v) + u^{-1}\alpha(u-1, v-1).$$

From (16)

$$P(V_n = v) = \sum_{u=v}^{n} \alpha(u, v) n^{-(u+1)}(n)_n u. \tag{17}$$

This distribution is derived in a more complicated form by Folkert (1955), employing the aforementioned Katz identity. Using (12) and (17), we may create an identity for $E(V_n)$. Similar identities can be developed for higher moments of V_n. Using a generating function argument (Riordan, 1958: p. 71), the w^{th} factorial moments of V_n, given U_n, can be derived:

$$E[(V_n)_w \mid U_n = u] = \begin{cases} \left.\dfrac{1}{u!}\dfrac{\partial^w [t(t+1)) \dots (t+u-1)]}{\partial t^w}\right|_{t=1}, & w \le u \\ 0 & , w > u. \end{cases}$$

At $w = 1$

$$E(V_n \mid U_n = u) = \sum_{r=1}^{u} \frac{1}{r}, \tag{18}$$

whence the conditional mean of V_n, given U_n, behaves like log U_n when U_n is large.

Similar computation reveals that the conditional variance of V_n, given U_n, also behaves like log U_n when U_n is large.

For the exact distribution of $B_{n,r}$ (equivalently, $A_{n,r}$), let

$$\beta(u, r, k) \equiv P(B_{n,r} = k \mid U_n = u). \tag{19}$$

It is straightforward to show that

$$\beta(u, r, k) = r^{-k}(k!)^{-1}\beta(u - kr, r, 0).$$

Since $\beta(w, r, 0) = 1 - \sum_{k=1}^{[w/r]} \beta(w, r, k)$ ([ ] indicates greater integer in) and since $\beta(w, r, 0) = 1$ when $w < r$, $\beta(u, r, k)$ can be computed recursively. Thus, from (19)

$$P(B_{n,r} = k) = \sum_{u=kr}^{n} \beta(u, r, k) n^{-(u+1)}(n)u^u. \tag{20}$$

From Riordan (1958, pp. 82–84) we may show that the w^{th} factorial moment of $B_{n,r}$, given $U_n = u$, is

$$E((B_{n,r})_w \mid U_n = u) = \begin{cases} r^{-w}, & w \le [u/r] \\ 0\ , & w > [w/r]. \end{cases} \tag{21}$$

At $w = 1$, $E(B_{n,r} \mid U_n = u) = r^{-1}$, $r \le u$, and summing both sides, $1 \le r \le u$ again yields (18). Var $(V_{n,r} \mid U_n = u) = r^{-1}$ as well, if $2r \le u$.

In concluding this section, we examine the expected length of a cycle denoted by ECL. We first compute the likelihood of any particular cycle structure under a random T. Let m_l be the number of cycles of length l, $l = 1,2,\ldots,n$, and let $m_0 = n - \Sigma m_l l =$ the number of transient states. Then for $\Sigma m_l l \le n$,

$$\begin{aligned} &P(m \text{ cycles of length } l,\ l = 1,2,\ldots,n, \text{ and } m_0 \text{ transient states}) \\ &\equiv P(m_0, m_1, \ldots, m_n) \\ &= P(m_1, \ldots, m_n \mid U_n = n - m_0)P(U_n = n - m_0) \\ &= \left(\prod_{l=1}^{n-m_0} m_l!\ \prod_{l=1}^{n-m_0} l^{m_l}\right)^{-1} n^{-(n-m_0+1)}(n)_{n-m_0}(n - m_0) \\ &= \left(\prod_{l=1}^{n-m_0} m_l!\ \prod_{l=1}^{n-m_0} l^{m_l}\right)^{-1} n!\, n^{-(n-m_0+1)}(n - m_0). \end{aligned}$$

(Note: Sherlock (1979) considers the conditional distribution above at greater length.)

For the cycle structure $(m_0, m_1, \ldots, m_n)$ the average cycle length will be $(\Sigma m_l)^{-1}\Sigma m_l l$, whence

$$ECL = \sum (\Sigma m_l)^{-1}(\Sigma m_l l)P(m_0, m_1, \ldots, m_n), \tag{22}$$

where the outer sum is over the set $\{(m_0, m_1, \ldots, m_n)\colon m_l \geq 0, \Sigma m_l l \leq n\}$.

More directly, since $\Sigma m_l l$ is a value of U_n and Σm_l is a value of V_n,

$$ECL = E[(V_n)^{-1}U_n].$$

It is important to note that in determining ECL we have, for a particular net, defined "average cycle length" assuming cycles to be equally likely, e.g. if a net has 3 cycles of lengths 10, 5 and 3 we obtain an average length $= 10\left(\frac{1}{3}\right) + 5\left(\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) = 6$. Average cycle length for a particular net may also be defined assuming equally likely selection of a cyclic state. For the above example we would then obtain an average cycle length $= 10\left(\frac{10}{18}\right) + 5\left(\frac{5}{18}\right) + 3\left(\frac{3}{18}\right) = 7.4$. Kauffman (1969a) and Cull (1978) studied ECL under this latter definition.

4. *Asymptotic Results.* Using Harris' idea (1960, p. 1047), we obtain the asymptotic probability density of U_n. Letting $W_n = U_n/\sqrt{n}$ and using (15), we may show that W_n converges in distribution to a random variable W having a Rayleigh distribution, i.e.

$$f_W(w) = w\, e^{-w^2/2}, \quad w > 0.$$

Hence $U_n \xrightarrow{p} \infty$. Since $E(W^r) = 2^{r/2}\Gamma(r + 2/2)$, $r > -2$, we have the asymptotic behavior of all moments of U_n, i.e. $E(U_n^r) = O(n^{r/2})$.

From remarks after (18), $E(V_n)$ behaves like $E(\log U_n)$ and $\operatorname{var}(V_n) = \operatorname{var} E(V_n | U_n) + E \operatorname{var}(V_n | U_n)$ behaves like $\operatorname{var}(\log U_n) + E(\log U_n)$. Since by simple expansions $E(\log U_n) = O(\log n)$ and $\operatorname{var}(\log U_n) = O(1)$, we have $E(V_n)$ and $\operatorname{var}(V_n)$ both $O(\log n)$.

For ECL, a bivariate expansion of $(V_n^{-1})U_n$ reveals that the $[E(V_n)]^{-1}E(U_n)$ term dominates and, thus, that $ECL = O[(\log n)^{-1}\sqrt{n}]$.

Finally, we show that the asymptotic distribution of $B_{n,r}$ is Poisson (r^{-1}). The well-known fact that if X is distributed Poisson (λ), then the w^{th} factorial moment of X is λ^w (see, e.g. Johnson and Kotz, 1969: p. 90) means we only need show that $\lim_{n\to\infty} E[(T_{n,r})_w] = r^{-w}$, $w = 1, 2, \ldots$. Using

(21) and the fact that $U_n \xrightarrow{p} \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} E[(B_{n,r})_w] &= \lim_{n\to\infty} E[E(B_{n,r})_w | U_n] \\ &= \lim_{n\to\infty} r^{-w} P[(U_n/r) \geq w] \\ &= r^{-w}.\end{aligned}$$

5. *Summary.* We summarize the results of the previous sections by returning to the completely random net, setting $n = 2^N$:

(i) the expected number of cyclic states is of the order $2^{N/2}$

(ii) the expected number of transient states is of the order 2^N

(iii) the expected number of cycles is of the order N

(iv) the likelihood that any particular state is cyclic is of the order $2^{-N/2}$

(v) the expected number of cycles of length r converges to $1/r$

(vi) the expected number of states on cycles of length r converges to 1

(vii) the expected cycle length is of the order $(N^{-1})2^{N/2}$.

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